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A sufficient condition for the sum of complemented subspaces to be complemented

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We provide a sufficient condition for the sum of a finite number of complemented subspaces of a Banach space to be complemented. Under this condition, the formula for a projection onto the sum is given. The condition is sharp (in a certain sense). As an application, we provide a sufficient condition for the complementability of the sum of marginal subspaces in L^p.

Keywords: *sum of subspaces, complemented subspace, closed subspace, marginal subspace, projection.*

1. Complemented subspaces in Banach spaces. Let X be a (complex or real) Banach space. By a subspace of X, we will mean a linear subset of X. Let M be a subspace of X. M is said to be complemented in X if there exists a continuous linear projection onto M, i.e., a continuous linear operator $P: X \to X$ such that $Px \in M$ for all $x \in X$ and $Px = x$ for $x \in M$. It is easily seen that each complemented subspace is closed. Note that one can give the following (equivalent) definition of complementability: a subspace M is said to be complemented in X if M is closed and there exists a closed subspace N (a complement) such that $M \cap N = \{0\}$ and $M + N = X$.

If X is a Hilbert space, then each closed subspace M of X is complemented in X (one can consider the orthogonal projection onto M). Of course, this is true if X is isomorphic to a Hilbert space. But if X is not isomorphic to a Hilbert space, then, by the Lindenstrauss—Tzafriri theorem, X contains a closed subspace which is not complemented in X.

For the further information on complemented and uncomplemented subspaces in Banach spaces and, in particular, various examples of uncomplemented closed subspaces see, e.g., [1, 2] and the references therein.

2. Formulations of problems. Let X be a Banach space and $X_1, \ldots, X_n$ be complemented subspaces of X. Define the sum of $X_1, \ldots, X_n$ in the natural way, namely,

$$X_1 + \ldots + X_n := \{x_1 + \ldots + x_n \mid x_1 \in X_1, \ldots, x_n \in X_n\}.$$

The natural question arises:

Question 1: Is $X_1+\ldots+X_n$ complemented in X?

Note that Question 1 makes sense, since the sum of two complemented subspaces may be uncomplemented and even nonclosed. A simple example is as follows: if X is a Hilbert space, then a subspace is complemented if and only if it is closed, and there are well-known simple examples of two closed subspaces with nonclosed sum.

If Question 1 has a positive answer, then the next natural question arises:

Question 2: Suppose that we know some (continuous linear) projections $P_1,\ldots,P_n$ onto $X_1,\ldots,X_n$, respectively. Is there a formula for a projection onto $X_1+\ldots+X_n$ (in terms of $P_1,\ldots,P_n$) (of course, under certain conditions)?

Since each complemented subspace is closed, Question 1 is closely related to the following

Question 3: Is $X_1+\ldots+X_n$ closed in X?

It is worth mentioning that if X is a Hilbert space, then Question 1 coincides with Question 3.

Systems of subspaces $X_1,\ldots,X_n$, for which Question 3 is very important, arise in various branches of mathematics, for example, in theoretical tomography and the theory of ridge functions (plane waves) (see, e.g., [3, Introduction, Chapter 7 and the references therein]), theory of wavelets and multiresolution analysis (see, e.g., [4] and references therein), statistics (see, e.g., [5]), approximation algorithms in Hilbert and Banach spaces and, in particular, methods of alternating projections (see, e.g., [3, Chapter 9 and the bibliography therein]) and others.

3. Linear independence. Another property of systems of subspaces, which will be of interest to us, is the linear independence of subspaces. A system of subspaces $X_1,\ldots,X_n$ is said to be linearly independent if the equality $x_1+\ldots+x_n=0$, where $x_1\in X_1,\ldots,x_n\in X_n$, implies that $x_1=\ldots=x_n=0$.

4. Notation. Throughout the paper, X is a real or complex Banach space with norm $\|\cdot\|$. The identity operator on X is denoted by I. By a projection we always mean a continuous linear projection. The kernel of an operator T will be denoted by $\ker(T)$. All vectors are vector-columns; the letter "*t*" means the transpose.

5. Known results. Let X be a Banach space, $X_1,\ldots,X_n$ be complemented subspaces of X, and $P_1,\ldots,P_n$ be projections onto $X_1,\ldots,X_n$, respectively.

For $n=2$ sufficient conditions for X_1+X_2 to be complemented in X can be found in [6— 9]. As an example, we present a result from [9]: if the restriction of the operator $I-P_2P_1$ to its invariant subspace X_2 is Fredholm, then X_1+X_2 is complemented in X. Concerning Question 2, a few formulas for a projection onto X_1+X_2 (under certain conditions) can be found in [7].

For arbitrary n each of the following conditions is sufficient for $X_1+\ldots+X_n$ to be complemented in X:

1. ([6, Corollary]) $X_1,\ldots,X_n$ are pairwise totally incomparable;

2. ([7, Corollary 2.9]) P_iP_j is compact for every pair $i\neq j$, $i,j\in\{1,\ldots,n\}$. Moreover, under this condition, there exists a projection P onto $X_1+\ldots+X_n$ such that P equals $P_1+\ldots+P_n$ modulo compact operators.

6. Our results. We will provide a new sufficient condition for $X_1+\ldots+X_n$ to be complemented in X. Under the condition, a formula for a projection onto the sum will be given.

We begin with a simple observation on Questions 1 and 2. The observation was used by many authors. If $P_i|_{X_j}=0$ for all $i\neq j$, $i,j\in\{1,\ldots,n\}$, then $X_1,\ldots,X_n$ are linearly independent, their sum is complemented in X, and $P=P_1+\ldots+P_n$ is a projection onto $X_1+\ldots+X_n$.

Our result can be regarded as a strengthening of the observation.

Suppose that nonnegative numbers ε_{ij}, $i \neq j$, $i, j \in \{1, \ldots, n\}$ are such that

$$\| P_i x \| \leqslant \varepsilon_{ij} \| x \|, \quad x \in X_j$$

for every $i \neq j$, $i, j \in \{1, \ldots, n\}$.

Define the $n \times n$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} 0, & if\ i = j; \\ \varepsilon_{ij}, & if\ i \neq j. \end{cases}$$

Denote, by $r(E)$, the spectral radius of E. Set $A := P_1 + \ldots + P_n$.

Theorem 1. *If $r(E) < 1$, then the subspaces $X_1, \ldots, X_n$ are linearly independent, their sum is complemented in X, and the subspace $\ker(P_1) \cap \ldots \cap \ker(P_n)$ is a complement* of $X_1 + \ldots + X_n$ *in X. Moreover, the sequence of operators*

$$I - (I - A)^N$$

converges uniformly to the projection P onto $X_1 + \ldots + X_n$ along $\ker(P_1) \cap \ldots \cap \ker(P_n)$ as $N \to \infty$.

For practical applications, it is important to know how rapidly the sequence $I - (I - A)^N$ converges to P. Our next result shows that the rate of convergence can be estimated from above by $C\alpha^N$, where $\alpha \in [0, 1)$. To formulate the result, we need the following notation: for two vectors $u, v \in \mathbb{R}^n$, we write $u \leqslant v$ if $u \leqslant v$ coordinatewise.

Theorem 2. The following statements on the rate of convergence of $I - (I - A)^N$ to P are true.

1. *Suppose a vector $w = (w_1, \ldots, w_n)^t$ with positive coordinates and a number $\alpha \in [0, 1)$ satisfy $Ew \leqslant \alpha w$. Then*

$$\| I - (I - A)^N - P \| \leqslant (w_1 + \ldots + w_n) \max\{(1/w_1) \| P_1 \|, \ldots, (1/w_n) \| P_n \|\} \frac{\alpha^N}{1 - \alpha}$$

for each $N \geqslant 1$.

2. *Suppose a vector $w = (w_1, \ldots, w_n)^t$ with positive coordinates and a number $\alpha \in [0, 1)$ satisfy $E^t w \leqslant \alpha w$. Then*

$$\| I - (I - A)^N - P \| \leqslant (w_1 \| P_1 \| + \ldots + w_n \| P_n \|) \max\{(1/w_1), \ldots, (1/w_n)\} \frac{\alpha^N}{1 - \alpha}$$

for each $N \geqslant 1$.

Remark 1. Since E is a nonnegative matrix, the existence of a vector $w \in \mathbb{R}^n$ with positive coordinates and a number $\alpha \in [0, 1)$ such that $Ew \leqslant \alpha w$ is equivalent to $r(E) < 1$. More precisely, if such w and α exist, then $r(E) \leqslant \alpha < 1$ (see [10, Corollary 8.1.29]). Conversely, suppose that $r(E) < 1$. If E is irreducible, then one can take α to be $r(E)$, and w is a Perron—Frobenius vector of E. If E is not irreducible, then we consider the matrix $E' = (e_{ij} + \delta)$ for sufficiently small $\delta > 0$, and take α to be $r(E')$ and w a Perron—Frobenius vector of E'.

Similarly, the existence of a vector w with positive coordinates and a number $\alpha \in [0, 1)$ such that $E^t w \leqslant \alpha w$ is equivalent to $r(E) < 1$.

The assumption $r(E)<1$ is a sharp sufficient condition for $X_1+\ldots+X_n$ to be complemented in X.

Theorem 3. *Let* $E=(e_{ij})$ *be an* $n\times n$ *matrix with* $e_{ii}=0$ for $i=1,\ldots,n$ *and* $e_{ij}\geqslant 0$ *for every pair* $i\neq j$, $i,j\in\{1,\ldots,n\}$. If $r(E)=1$, *then there exist a Banach space* X, *complemented subspaces* $X_1,\ldots,X_n$ *of* X, *and projections* $P_1,\ldots,P_n$ *onto* $X_1,\ldots,X_n$, *respectively, such that*

1. $\|P_i x\|=e_{ij}\|x\|$, $x\in X_j$, *for each pair* $i\neq j$, $i,j\in\{1,\ldots,n\}$;

2. $X_1,\ldots,X_n$ *are linearly independent*;

3. $X_1+\ldots+X_n$ *is closed and not complemented in* X.

Remark 2. In the case where $r(E)>1$ the theorem can be applied to the matrix $(1/r(E))E$.

7. Sums of marginal subspaces. As an application of Theorem 1, we provide a sufficient condition for the complementability of the sum of marginal subspaces in L^p.

Let $(\Omega,\mathcal{F},\mu)$ be a probability space. Denote by $\mathbb{K}$ a base field of scalars, i.e., $\mathbb{R}$ or $\mathbb{C}$. For an $\mathcal{F}$-measurable function (random variable) $\xi:\Omega\to\mathbb{K}$ we denote by $E\xi$ the expectation of ξ (if it exists). Two random variables ξ and η are said to be equivalent if $\xi(\omega)=\eta(\omega)$ for μ-almost all ω. For $p\in[1,\infty)\cup\{\infty\}$ denote by $L^p(\mathcal{F})=L^p(\Omega,\mathcal{F},\mu)$ the set of equivalence classes of random variables $\xi:\Omega\to\mathbb{K}$ such that $E|\xi|^p<\infty$ if $p\in[1,\infty)$, and ξ is μ-essentially bounded if $p=\infty$. For $\xi\in L^p(\mathcal{F})$, set $\|\xi\|_p=(E|\xi|^p)^{1/p}$ if $p\in[1,\infty)$ and $\|\xi\|_\infty=\operatorname{esssup}|\xi|$ if $p=\infty$. Then $L^p(\mathcal{F})$ is a Banach space. For every sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$, we define the marginal subspace corresponding to $\mathcal{A}$, $L^p(\mathcal{A})$, as follows. $L^p(\mathcal{A})$ consists of elements (equivalence classes) of $L^p(\mathcal{F})$ which contain at least one $\mathcal{A}$-measurable random variable. Denote by $L^p_0(\mathcal{A})$ the subspace of all $\xi\in L^p(\mathcal{A})$ with $E\xi=0$.

We study the following problem. Let $\mathcal{F}_1,\ldots,\mathcal{F}_n$ be sub-σ-algebras of $\mathcal{F}$. Question: when is the sum of the corresponding marginal subspaces, $L^p(\mathcal{F}_1)+\ldots+L^p(\mathcal{F}_n)$, complemented in $L^p(\mathcal{F})$? One can check that $L^p(\mathcal{F}_1)+\ldots+L^p(\mathcal{F}_n)$ is complemented in $L^p(\mathcal{F})$ if and only if $L^p_0(\mathcal{F}_1)+\ldots+L^p_0(\mathcal{F}_n)$ is. Since each complemented subspace is closed, the question on the complementability of the sum of marginal subspaces is closely related to the question on the closedness of the sum (for $p=2$, these questions coincide). One can check that $L^p(\mathcal{F}_1)+\ldots+L^p(\mathcal{F}_n)$ is closed in $L^p(\mathcal{F})$ if and only if $L^p_0(\mathcal{F}_1)+\ldots+L^p_0(\mathcal{F}_n)$ is.

The question on the closedness of the sum of marginal subspaces arises, for example, in additive modeling (see, e.g., [11, Subsection 8.1]) and the theory of ridge functions (see, e.g., [3, Chapter 7]) (note that every subspace of ridge functions $L^p(a;K)$ can be considered as marginal).

The question on the closedness is not trivial; examples where $L^p(\mathcal{F}_1)+L^p(\mathcal{F}_2)$ is not closed in $L^p(\mathcal{F})$ can be found in [12, Proposition 4.4(a)] (for $p\in[1,\infty)$), [11, Subsection 8.3] (for $p=2$), [3, Section 7.2] (for $p\in[1,\infty)\cup\{\infty\}$).

Sufficient conditions for the sum of marginal subspaces to be closed can be found in [5, p.1332, Proof of Lemma 1], [11, Section 8] and [3, Chapter 7]. Our result (Theorem 4) is motivated by the result of [5] and contains it as a special case.

To formulate our result on the complementability of the sum of marginal subspaces, we need an auxiliary notion. Let $(\Omega,\mathcal{F},\mu)$ be a probability space. For two sub-σ-algebras $\mathcal{A},\mathcal{B}$ of $\mathcal{F}$ define the following measure of their dependence:

$$\psi'(\mathcal{A},\mathcal{B})=\inf\left\{\frac{\mu(A\cap B)}{\mu(A)\mu(B)}\,\middle|\,A\in\mathcal{A},B\in\mathcal{B},\mu(A)>0,\mu(B)>0\right\}.$$

This measure of dependence is well known (see, e.g., [13]). It is easily seen that $0 \leqslant \psi'(\mathcal{A},\mathcal{B}) \leqslant 1$ and $\psi'(\mathcal{A},\mathcal{B})=1$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are independent.

Let us formulate our result. Let $\mathcal{F}_1,\ldots,\mathcal{F}_n$ be sub-σ-algebras of $\mathcal{F}$. Define the $n\times n$ matrix $E=(e_{ij})$by

$$e_{ij}=\begin{cases} 0, & if \;\; i=j; \\ 1-\psi'(\mathcal{F}_i,\mathcal{F}_j), & if \;\; i\neq j. \end{cases}$$

Theorem 4. If $r(E)<1$, then the marginal subspaces $L_0^p(\mathcal{F}_1),\ldots,L_0^p(\mathcal{F}_n)$ are linearly independent and their sum is complemented in $L^p(\mathcal{F})$ (for arbitrary $p\in[1,\infty)\cup\{\infty\}$).

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ДОСТАТНЯ УМОВА ДЛЯ ТОГО, ЩОБ СУМА ДОПОВНЮВАЛЬНИХ ПІДПРОСТОРІВ БУЛА ДОПОВНЮВАЛЬНОЮ

Наведено достатню умову для того, щоб сума скінченного числа доповнювальних підпросторів банахового простору була доповнювальною. За цієї умови отримано формулу для проектора на цю суму підпросторів. Ця умова є точною (в певному сенсі). Як застосування наведено достатню умову для доповнювальності суми маргінальних підпросторів у просторі L^p.

Ключові слова: *сума підпросторів, доповнювальний підпростір, замкнений підпростір, маргінальний підпростір, проектор.*

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ДОСТАТОЧНОЕ УСЛОВИЕ ДЛЯ ТОГО, ЧТОБЫ СУММА ДОПОЛНЯЕМЫХ ПОДПРОСТРАНСТВ БЫЛА ДОПОЛНЯЕМА

Приведено достаточное условие для того, чтобы сумма конечного числа дополняемых подпространств банахова пространства была дополняема. При этом условии получена формула для проектора на эту сумму подпространств. Это условие является точным (в определенном смысле). В качестве применения приведено достаточное условие для дополняемости суммы маргинальных подпространств в пространстве L^p.

Ключевые слова: *сумма подпространств, дополняемое подпространство, замкнутое подпространство, маргинальное подпространство, проектор.*