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On induced modules over locally Abelian-by-polycyclic groups of finite rank

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We develop some methods for studying the modules over group rings, which are based on properties of induced modules and on the embedding of these modules in the modules over rings of quotients of group rings. Using these methods, we have obtained the criteria of primitivity for group algebras of certain classes of locally soluble groups.

Keywords: *induced modules, group rings, particle rings, primitive algebras.*

Let R be a ring, and let M, X, and Y be R-modules. We say that X and Y are separated by M if X and Y have no non-zero isomorphic R-sections, which are isomorphic to a submodule of M.

Let A be a normal subgroup of a group H, let k be a field, and let M and W be kA-modules. Then the subgroup $\mathrm{Sep}_{(H,A)}(M,W)$ of H generated by all elements $g \in H$ such that W and Wg are not separated by M is said to be the separator of W in H.

Let H be a subgroup of the group G. The subgroup H is said to be dense in G, if, for any $g \in G$, there is an integer $n \in \mathbb{N}$ such that $g^n \in H$. Let G be a locally Abelian-by-polycyclic-by-finite group of finite rank, and let H be a finitely generated dense subgroup of G. It follows from Lemma 2.1.3 of [1] that H has an Abelian normal torsion-free subgroup A such that the quotient group H/A is polycyclic and A has no infinite polycyclic G-sections. The pair (H, A) will be called an important pair of G.

Let G be a locally Abelian-by-polycyclic-by-finite group of finite rank, let k be a field, and let M be a kG-module. Let $0 \neq a \in M$. Then the subgroup $\mathrm{Sep}_{(H,A)}(M, akA)$ generated by subgroups $\mathrm{Sep}_{(H,A)}(M, akA)$, where (H, A) runs through all important pairs of G, is called the separator of a in G.

Theorem 1. *Let G be a locally Abelian-by-policyclyc-by-finite group of finite rank, let J be a principal ideal domain of zero characteristic, and let M be a J-torsion-free JG-module. Then there is an element $a \in M \setminus \{0\}$ such that:*

(*i*) $aJG = aJS \otimes_{JS} JG$, *where* $S = \mathrm{Sep}_G(M', a)$, $M' = M \otimes_J k$ *and* k *is the field of fractions of* J;

(ii) *either* $r_0(S) < r_0(G)$ *or, for any finitely generated dense subgroup* H *of* S, *there is an important pair* (H, A) *of* S *such that the quotient group* $\tilde{A} = A / C_A(aJH)$ *is torsion-free and* aJH *is* $\tilde{J}A$ *-torsion-free.*

A group ring RG is said to be locally Noetherian, if, for any finitely generated subgroup H of G, the group ring RH is Noetherian. Group rings of locally polycyclic-by-finite groups over a Noetherian ring give us an obvious example of locally Noetherian group rings. "Local approach" for studying the modules over locally Noetherian group rings was developed in [2—4]. In [5, 6], we modernized this approach by passing to the rings of quotients of group rings striving to make it available for studying not only locally Noetherian group rings. The following proposition shows how we can obtain a Noetherian ring by passing to an appropriate ring of quotients of the group ring of an Abelian-by-polycyclic-by-finite group.

Proposition 1. *Suppose that a group* H *has an Abelian normal torsion-free subgroup* A *of finite rank such that the quotient group* H/A *is polycyclic-by-finite. Let* J *be a principal ideal domain, let* $\sigma \in J$ *be a prime element, and let* $Q_\sigma = JA \setminus \sigma JA$. *Let* M *be a* JH *-module, which is* JA *-torsion-free. Then:*

(i) *the ring of quotients* $JH(Q_\sigma)^{-1}$ *exists and is Noetherian*;

(ii) $M \leqslant M \bigotimes_{JA} JA(Q_\sigma)^{-1} = M(Q_\sigma)^{-1}$ *and* $M(Q_\sigma)^{-1}$ *is a* $JH(Q_\sigma)^{-1}$*-module.*

In the following proposition, we use the modernized "local approach" for studying the properties of divisibility in modules over group rings of locally Abelian-by-polycyclic-by-finite groups, which are not locally Noetherian in general.

Proposition 2. *Let* S *be a locally Abelian-by-policyclyc-by-finite group of finite rank, let* J *be a principal ideal domain of zero characteristic, and let* $M = aJS$ *be a* J *-torsion-free* JS *-module such that, for any finitely generated dense subgroup* H *of* S, *there is an important pair* (H, A) *of* S *such that the quotient group* $\tilde{A} = A / C_A(aJH)$ *is torsion-free and* aJH *is* $\tilde{J}A$ *-torsion-free. Suppose that, for some prime element* $\sigma \in J$, *the module* aJS *is* σ *-divisible (i.e.* $aJS\sigma = aJS$ *). Then, for any finitely generated dense subgroup* H *of* S, *there is an important pair* (H, A) *of* S *such that the quotient group* $\tilde{A} = A / C_A(aJH)$ *is torsion-free, the* $\tilde{J}A$ *-module* aJH *is* $\tilde{J}A$ *-torsion-free, and:*

(i) $aJH \leqslant aJH \bigotimes_{JA} JA(Q_\sigma)^{-1} = aJH(Q_\sigma)^{-1}$ *and* $aJH(Q_\sigma)^{-1}$ *is an* σ*-divisible* $J\hat{H}(Q_\sigma)^{-1}$*-module, where* $\hat{H} = H / C_A(aJH)$ *and* $Q_\sigma = J\hat{A} \setminus \sigma J\hat{A}$;

(ii) *there exist elements* $\beta \in Q_\sigma$ *and* $b \in J\hat{H}$ *such that* $\sigma b - \beta \in Ann_{J\hat{H}}(a)$.

A combination of Theorem 1 and Proposition 2 gives us the following result.

Theorem 2. *Let* G *be a locally Abelian-by-policyclyc-by-finite group of finite rank and let* J *be a principal ideal domain of zero characteristic. If* M *is an irreducible* JG *-module, then* $M\sigma = 0$ *for some prime element* $\sigma \in J$.

If a group Γ acts on a set A, we say an element is (Γ)*-orbital*, if its orbit is finite, and write $\Delta_\Gamma(A)$ for the subset of such elements. The FC-center of a group G, denoted by $\Delta(G)$, is just $\Delta_G(G)$, where the action of G on itself is the conjugation. In [7], Farkas and Pasman proved that the group algebra kG of a polycyclic-by-finite group G over a field k of zero characteristic is primitive, if and only if the FC-center $\Delta(G)$ of G is trivial and conjectured that the triviality of $\Delta(G)$ is also the necessary and sufficient condition for the primitivity of the group algebra $\Delta(G)$ of a polycyclic-by-finite group G over an arbitrary not locally finite field k. This assertion was proved by Roseblade in [8]. In [9], Brown conjectured that this result remains true in the case of a soluble group G of finite rank, and Brookes [10] proved that if G is a soluble group of finite rank such that $\Delta(G)=1$ and a field k is not locally finite, then kG is primitive. However, it is still unknown whether the triviality of $\Delta(G)$ is also necessary for the primitivity of kG. Theorem 2 allows us to prove the following.

Theorem 3. *Let G be a locally Abelian-by-policyclyc group of finite rank and let k be a field of characteristics zero. The group algebra kG is primitive, if and only if $\Delta(G) = 1$.*

Let G be a group and let k be a field. We say that the Nullstellensatz holds for kG, if $\mathrm{End}_{kG}M$ is algebraic over k for any irreducible kG-module M (see [4]). Various versions of the Nullstellensatz were proved for group rings of polycyclic, locally finite and locally polycyclic-by-finite groups by Hall [11], McConnell [12], Baer [2], and Brown [4]. Due to Theorem 2, we can obtain the following result.

Theorem 4. *Let G be a locally Abelian-by-policyclyc group of finite rank, let k be a field of zero characteristic and let M be an irreducible kG-module. Then $\mathrm{End}_{kG}M$ is algebraic over k.*

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ПРО ІНДУКОВАНІ МОДУЛІ НАД ГРУПАМИ СКІНЧЕННОГО РАНГУ, СКІНЧЕННО ПОРОДЖЕНІ ПІДГРУПИ ЯКИХ Є РОЗШИРЕННЯМИ АБЕЛЕВИХ ГРУП ЗА ДОПОМОГОЮ ПОЛІЦИКЛІЧНИХ

У роботі розвиваються методи вивчення модулів над груповими кільцями, які базуються на властивостях індукованих модулів і на вкладенні цих модулів у модулі над кільцями часток групових кілець. За допомогою цих методів, зокрема, отримано критерії примітивності групових алгебр деяких класів локально розв'язних груп.

Ключові слова: *індуковані модулі, групові кільця, кільця часток, примітивні алгебри.*

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ОБ ИНДУЦИРОВАННЫХ МОДУЛЯХ НАД ГРУППАМИ КОНЕЧНОГО РАНГА, КОНЕЧНО ПОРОЖДЕННЫЕ ПОДГРУППЫ КОТОРЫХ ЯВЛЯЮТСЯ РАСШИРЕНИЯМИ АБЕЛЕВЫХ ГРУПП С ПОМОЩЬЮ ПОЛИЦИКЛИЧЕСКИХ

В работе развиваются методы изучения модулей над групповыми кольцами, которые базируются на индуцированности модулей и на вложении этих модулей в модули над кольцами частных групповых колец. С помощью этих методов, в частности, получены критерии примитивности групповых алгебр некоторых классов локально разрешимых групп.

Ключевые слова: *индуцированные модули, групповые кольца, кольца частиц, примитивные алгебры.*