

<https://doi.org/10.15407/dopovidi2021.06.012>

UDC 512.544

T.V. Velychko, <https://orcid.org/0000-0001-9830-5512>

Oles Honchar Dnipro National University

E-mail: etativ27@gmail.com

On the structure of some non-periodic groups whose subgroups of infinite special rank are transitively normal

Presented by Corresponding Member of the NAS of Ukraine V.P. Motornyi

A group G has a finite special rank r , if every finitely generated subgroup of G can be generated by at most r elements, and there exists a finitely generated subgroup H which has exactly r generators. This paper is devoted to generalized radical non-Abelian groups of infinite special rank whose subgroups of infinite special rank are transitively normal.

Keywords: *finite special rank, periodic group, locally nilpotent radical, transitively normal subgroups.*

The groups with certain prescribed properties of subgroups are among central research subjects in group theory. The study of such groups led to the emergence of many concepts such as the finiteness conditions, local nilpotency, local solubility, subnormality, permutability, some important numerical invariants of groups (as, e.g., distinct group ranks), and others. Choosing the specific prescribed properties and the definite families of subgroups that possess these properties, we come to the distinct classes of groups.

In papers [1, 2], the study of groups whose subgroups of infinite special rank are transitively normal was initiated.

A group G has a *finite special rank* r , if every finitely generated subgroup of G is generated by at most r elements, and r is the least integer with this property. If there is no such r , then we say that G has an *infinite special rank* [3]. The theory of the groups of finite special rank is one of the most developed branches of group theory (see, e.g., [4–7]). In [8], M.R. Dixon, M.J. Evans, and H. Smith initiated the investigation of the groups whose subgroups of infinite special rank have some fixed property P . These investigations were continued by many authors for various properties P (see, e.g., [5]).

Цитування: Velychko T.V. On the structure of some non-periodic groups whose subgroups of infinite special rank are transitively normal. *Допов. Нац. акад. наук Укр.* 2021. № 6. С. 12–14.
<https://doi.org/10.15407/dopovidi2021.06.012>

A subgroup H of the group G is *transitively normal* in G , if H is normal in every subgroup $K \geq H$ in which H is subnormal [9]. There are many natural types of subgroups that are transitively normal. For example, pronormal subgroups and their generalizations are transitively normal [10].

In [1, 2], the study of the groups in which every subgroup of infinite special rank is transitively normal was begun. In [2], the structure of periodic soluble groups of infinite special rank with this property was described.

A group G is called *radical*, if G has an ascending series whose factors are locally nilpotent. If G is a radical group, then a locally nilpotent radical of G is non-trivial. It follows that a radical group has ascending series of normal subgroups whose factors are locally nilpotent. The factor A_{j+1}/A_j is called *G -eccentric*, if $C_G(A_{j+1}/A_j) \neq G$.

A group G is called *generalized radical*, if G has an ascending series whose factors are locally nilpotent or locally finite. If G is a generalized radical group, then either a locally nilpotent radical of G is non-trivial, or a locally finite radical of G is non-trivial. Therefore, a generalized radical group has an ascending series of normal subgroups whose factors are locally nilpotent or locally finite.

In [1], some non-periodic groups in which every subgroup of infinite special rank is transitively normal were studied. More precisely, the author proved that if G is a non-periodic locally generalized radical group with this property and G includes an ascendant locally nilpotent subgroup of infinite special rank, then G is Abelian. In the present paper, we continue the study of such groups with some additional restrictions on the locally nilpotent radical.

Lemma. *Let G be a generalized radical non-Abelian group of infinite special rank whose subgroups of infinite special rank are transitively normal. Suppose that $\text{Tor}(G) = \langle 1 \rangle$, and locally nilpotent radical L of G is Abelian. Then L has a G -invariant pure subgroup A , possessing a finite series*

$$\langle 1 \rangle = A_0 \leq A_1 \leq \dots \leq A_j \leq A_{j+1} \leq \dots \leq A_n = A$$

of G -invariant subgroups, satisfying the following conditions:

- (i) every subgroup A_j is pure in L , $1 \leq j \leq n$;
- (ii) every factor A_{j+1}/A_j is G -chief and G -eccentric, $0 \leq j \leq n-1$.
- (iii) a factor-group G/A is Abelian and has a finite periodic part.

Theorem. *Let G be a generalized radical non-Abelian group of infinite special rank whose subgroups of infinite special rank are transitively normal. Suppose that $\text{Tor}(G) = \langle 1 \rangle$, and locally nilpotent radical L of G is Abelian. Then the following assertions hold:*

- (i) L has a G -invariant pure subgroup A , possessing finite series

$$\langle 1 \rangle = A_0 \leq A_1 \leq \dots \leq A_j \leq A_{j+1} \leq \dots \leq A_n = A$$

of G -invariant pure subgroups whose factors A_{j+1}/A_j are G -chief and G -eccentric $0 \leq j \leq n-1$;

- (ii) $G = AC$ for some subgroup C such that $A \cap C = \langle 1 \rangle$, and every complement to A in G is conjugate to C .

- (iii) $C = S \times T$, where S is a free Abelian subgroup, having infinite 0-rank, and T is a finite Abelian subgroup.

REFERENCES

1. Kurdachenko, L. A., Subbotin, I. Ya. & Velychko, T. V. (2020). On the non-periodic groups, whose subgroups of infinite special rank are transitively normal. *Algebra Discrete Math.*, 29, No. 1, p. 74-84. <https://doi.org/10.12958/adm1357>
2. Semko, N. N. & Velychko, T. V. (2017). On the groups whose subgroups of infinite special rank are transitively normal. *Algebra Discrete Math.*, 24, No. 1, p. 34-45.
3. Maltsev, A. I. (1948). On groups of finite rank. *Mat. Sbornik*, 22, p. 351-352 (in Russian).
4. Dixon, M. R. (2008). Certain rank conditions on groups. *Note Mat.*, 28, No. 2, p. 151-175. <https://doi.org/10.1285/i15900932v28n2supplp151>
5. Dixon, M. R., Kurdachenko, L. A., Pypka, A. A. & Subbotin, I. Ya. (2016). Groups satisfying certain rank conditions. *Algebra Discrete Math.*, 22, No. 2, p. 184-200.
6. Dixon, M. R., Kurdachenko, L. A. & Subbotin, I. Ya. (2007). On various rank conditions in infinite groups. *Algebra Discrete Math.*, 6, No. 4, p. 23-43.
7. Dixon, M. R., Kurdachenko, L. A. & Subbotin, I. Ya. (2017). *Ranks of groups. The tools, characteristics and restrictions.* New York: Wiley.
8. Dixon, M. R., Evans, M. J. & Smith, H. (1997). Locally (soluble-by-finite) groups with all proper insoluble subgroups of finite rank. *Arch. Math.*, 68, p. 100-109. <https://doi.org/10.1007/s000130050037>
9. Kurdachenko, L. A. & Subbotin, I. Ya. (2006). Transitivity of normality and pronormal subgroups. In *Combinatorial group theory, discrete groups, and number theory. Contemporary Mathematics, Vol. 421* (p. 201-212). Providence, RI: Amer. Math. Soc.
10. Kirichenko, V. V., Kurdachenko, L. A. & Subbotin, I. Ya. (2011). Some related to pronormality subgroup families and the properties of a group. *Algebra Discrete Math.*, 11, No. 1, p. 75-108.

Received 05.04.2021

T.V. Величко, <https://orcid.org/0000-0001-9830-5512>

Дніпровський національний університет ім. Олеся Гончара
E-mail: etativ27@gmail.com

СТРУКТУРА ДЕЯКИХ НЕПЕРІОДИЧНИХ ГРУП,
В ЯКИХ ПІДГРУПИ НЕСКІНЧЕННОГО СПЕЦІАЛЬНОГО РАНГУ
Є ТРАНЗИТИВНО НОРМАЛЬНИМИ

Група G має скінченний спеціальний ранг r , якщо кожна скінченно породжена підгрупа G може бути породжена не більш ніж r елементами та існує скінченно породжена підгрупа H , яка має точно r породжуючих елементів. У статті наведено опис узагальнених радикальних неабелевих груп, в яких підгрупи нескінченного спеціального рангу є транзитивно нормальними.

Ключові слова: нескінченний спеціальний ранг, розв'язна група, періодична група, локально нільпотентний радикал, транзитивно нормальні підгрупи.