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V.Ya. Gutlyanskiĭ [1], https://orcid.org/0000-0002-8691-4617
O.V. Nesmelova [1, 2], https://orcid.org/0000-0003-2542-5980
V.I. Ryazanov [1, 3], https://orcid.org/0000-0002-4503-4939
A.S. Yefimushkin [1]

[1] Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slov'yansk
[2] Donbas State Pedagogical University, Slov'yansk
[3] Bogdan Khmelnytsky National University of Cherkasy
E-mail: vgutlyanskii@gmail.com, star-o@ukr.net, vl.ryazanov1@gmail.com, a.yefimushkin@gmail.com

Hilbert problem with measurable data for semilinear equations of the Vekua type

Presented by Corresponding Member of the NAS of Ukraine V.Ya. Gutlyanskiĭ

We prove the existence of solutions for the Hilbert boundary-value problem with arbitrary measurable data for the nonlinear equations of the Vekua's type $\partial_{\bar z} f(z) = h(z)q(f(z))$. The found solutions differ from the classical ones, because our approach is based on the notion of boundary values in the sense of angular limits along nontangential paths. The results obtained can be applied to the establishment of existence theorems for the Poincaré and Neumann boundary-value problems for the nonlinear Poisson equations of the form $\Delta U(z) = H(z)Q(U(z))$ with arbitrary measurable boundary data with respect to the logarithmic capacity. They can be also applied to the study of some semilinear equations of mathematical physics modeling such processes as the diffusion with absorption, plasma states, stationary burning etc. in anisotropic and inhomogeneous media.

Keywords: *Hilbert boundary-value problem, measurable boundary data, logarithmic capacity, semilinear equations of the Vekua type, nonlinear sources, angular limits, nontangent paths.*

1. On completely continuous Hilbert operators. The basic part of definitions and historic comments can be found in papers [1] and [2]. However, let us recall here that a completely continuous mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact ones of M_2, i.e., with compact closures in M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be compact on M_1.

The notion of completely continuous (compact) operators is due essentially, in the simplest partial cases, to Hilbert and F. Riesz, see the corresponding comments of Section VI.12 in [3], and to Leray and Schauder in the general case (see paper [4]).

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In paper [1], we considered *generalized analytic functions f with sources* $g \in L^p$, $p > 2$, that have generalized first derivatives by Sobolev and satisfy the equation

$$\frac{\partial f}{\partial \bar{z}} = g, \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right), \quad z = x + iy \tag{1}$$

and studied for them the Hilbert boundary-value problem under arbitrary boundary data that are measurable with respect to the logarithmic capacity.

In particular, Theorem 1 in [1] stated, for $\lambda : \partial\mathbb{D} \to \mathbb{C}, |\lambda(\zeta)| \equiv 1$, with countable bounded variation, $\varphi : \partial\mathbb{D} \to \mathbb{R}$ which is measurable with respect to the logarithmic capacity and $g : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, there exist generalized analytic functions $f : \mathbb{D} \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z \to \zeta} \operatorname{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}. \tag{2}$$

Furthermore, the space of such functions *f* has the infinite dimension.

Thus, the Hilbert boundary-value problem always has many solutions in the given sense for each such coefficient λ, boundary data φ and source *g*. Of course, the axiom of choice by Zermelo makes it possible to choose one of such correspondences named further as a Hilbert operator, but the latter with such a random choice can be completely discontinuous. Later on, to apply the approach of Leray—Schauder for extending Theorem 1 in [5] to the generalized analytic functions, satisfying nonlinear equations of the Vekua type, we need just the complete continuity of such correspondence.

Now, we show that, fixing only one antiderivative Φ for the function φ from Lemma 1 in [2], it is possible to obtain a completely continuous Hilbert operator. For this purpose, let us analyze the construction of solutions of Eq. (1) with the Hilbert boundary condition (2) from the proof of Theorem 1 in [1].

There, we often applied the *logarithmic (Newtonian) potential* N_G *of sources* $G \in L^p(\mathbb{C})$, $p > 2$, with compact supports given by the formula:

$$N_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w| G(w) dm(w). \tag{3}$$

However, by the linearity of the operator N_G with respect to *G*, we extend here definition (3) in a natural way to the complex-valued sources *G*, as usual, interpreting the imaginary parts of *G* as distributed currents. Recall also that, by Lemma 3 in [6], see also Theorem 2 in [7], $N_G \in W^{2,p}_{\text{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{C})$ with $\alpha := (p-2)/p$ and $\Delta N_G = G$ a.e.

Let us consider Eq. (1) in the unit disk $\mathbb{D}$. Extending *g* by zero outside of $\mathbb{D}$ and setting $P = N_G$ with $G = 2g$, $U = P_x$ and $V = -P_y$, we have that

$$H := U + iV = 2 \cdot \frac{\partial P}{\partial z}, \quad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right), \quad z = x + iy, \tag{4}$$

is just a generalized analytic function with the source g because the Laplacian

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\frac{\partial^2}{\partial \bar{z} \partial z}. \tag{5}$$

Note also, by the way, the connection of H with the known Pompeiu integral operator (see, e.g., relation (2.21) in [8]), that gives its representation in the explicit form

$$H(z) = T_g(z) := \frac{1}{\pi}\int_{\mathbb{C}} g(w)\frac{dm(w)}{z - w}. \tag{6}$$

Remark 1. In view of relation (6), we have by Theorem 1.19 in [9] that

$$|H(z)| \leqslant M_1 \| g \|_p \qquad \forall\, z \in \mathbb{C}, \tag{7}$$

$$|H(z_1) - H(z_2)| \leqslant M_2 \| g \|_p |z_1 - z_2|^{\alpha} \qquad \forall\, z_1, z_2 \in \mathbb{C}, \tag{8}$$

where the constants M_1 and M_2 depend only on $p > 2$, and $\alpha = (p-2)/p$. Thus, the linear operator H is completely continuous on compact sets in $\mathbb{C}$ and, in particular, on $\overline{\mathbb{D}}$ by the Arzela—Ascoli theorem (see, e.g. Theorem IV.6.7 in [3]).

Next, since $H \in C^{\alpha}_{\mathrm{loc}}(\mathbb{C})$, $\alpha := (p-2)/p$, the boundary function

$$\varphi_g(\zeta) := \lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot H(z)\} = \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot H(\zeta)\}, \quad \forall \zeta \in \partial\mathbb{D}, \tag{9}$$

is measurable with respect to the logarithmic capacity.

Consequently, the generalized analytic functions f with the source g satisfying the Hilbert condition (2) can be got as the sums $f = H + C$ with analytic functions C satisfying, in the sense of angular limits, the Hilbert boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot C(z)\} = \psi(\zeta) := \varphi(\zeta) - \varphi_g(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}. \tag{10}$$

In turn, by the construction of Theorem 5.1 in [10], such analytic functions C can be obtained as the products of 2 analytic functions A and B. The first

$$A(z) = e^{ia(z)}, \quad a(z) := \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \alpha_\lambda(\zeta)\frac{z+\zeta}{z-\zeta}\frac{d\zeta}{\zeta}, \qquad z \in \mathbb{D}, \tag{11}$$

with a function α_λ that is measurable with respect to the logarithmic capacity, bounded on $\partial\mathbb{D}$, of countable bounded variation and such that

$$\lambda(\zeta) = e^{i\alpha_\lambda(\zeta)} \qquad \text{q.e. on } \partial\mathbb{D}. \tag{12}$$

By Lemma 4.1 in [10], the angular limits of $\operatorname{Im} a(z)$ as $z \to \zeta$ q.e. on $\partial\mathbb{D}$ form a function $\beta : \partial\mathbb{D} \to \mathbb{R}$ that is measurable with respect to the logarithmic capacity. By Remark 2 in [2], the second analytic function B can be obtained in the form

$$B(z) = \mathbb{S}_{\Psi}(z) := \frac{z}{\pi} \int_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2} d\zeta, \quad z \in \mathbb{D}, \tag{13}$$

where Ψ is an antiderivative of the function ψe^{β} from Lemma 1 in [2].

Thus, analytic functions C can be represented in the more convenient form

$$C(z) = A(z) \cdot [\mathbb{S}_{\Phi}(z) - \mathbb{S}_{\Phi_g}(z)], \tag{14}$$

where Φ and Φ_g are antiderivatives of the functions of φe^{β} and $\varphi_* e^{\beta}$ from Lemma 1 in [2], respectively. Note that the analytic functions A and $\mathbb{S}_{\Phi}$ do not depend on the sources g at all. Let us choose the function Φ_g in a suitable way.

From this point on, we demand that all sources g have compact supports in the unit disk and belong to a disk $\mathbb{D}_{\rho} := \{z \in \mathbb{C} : |z| \leqslant \rho\}$ with a radius $\rho \in (0,1)$. Then the function $H(z)$, $z \in \mathbb{C}$, is analytic in a neighborhood of the unit circle $\partial\mathbb{D}$ and, in particular, $H(\zeta)$ is continuously differentiable with resspect to the variable ϑ, $\zeta = e^{i\vartheta}$, $\vartheta \in \mathbb{R}$. Moreover, by relation (6), we have that

$$H_{\vartheta}(\zeta) = i\zeta H'(\zeta) = \frac{\zeta}{\pi i} \int_{\mathbb{D}_{\rho}} g(w) \frac{dm(w)}{(\zeta - w)^2} \qquad \forall \zeta \in \partial\mathbb{D}. \tag{15}$$

Let us denote, by Λ, an antiderivative for the function $\overline{\lambda} e^{\beta}$ from Lemma 1 in [2].

Then the following function Φ_g is an antiderivative for the function $\varphi_g e^{\beta}$:

$$\Phi_g(\zeta) := \operatorname{Re}\{\Lambda(\zeta) H(\zeta) - \int_0^{\vartheta} \Lambda(\xi) H_{\theta}(\xi) d\theta + S(\vartheta)\}, \tag{16}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is a function of the form

$$S(\vartheta) := C(\vartheta) \int_0^{2\pi} \Lambda(\xi) H_{\theta}(\xi) d\theta, \quad \zeta = e^{i\vartheta}, \quad \xi = e^{i\theta}, \quad \vartheta, \theta \in [0, 2\pi], \tag{17}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ q.e. on $[0, 2\pi]$. Recall that the existence of such functions C follows from Lemma 3.1 in [11].

Let us show that the Hilbert operator H_g generated by the sums $H + C$ under the given choice of Φ and Φ_g in (14) is completely continuous on compact sets in $\mathbb{D}$. Recall that the analytic functions A and $\mathbb{S}_{\Phi}$ in representation (14) of C do not depend on the sources g. Hence, by Remark 1, it remains to show that the linear operator $\mathbb{S}_{\Phi_g}$ is completely continuous.

Indeed, by the construction of Φ_g in (16) and relations (6) and (15)

$$|\Phi_g(\zeta)| \leqslant \frac{1}{\pi}\frac{\|g\|_1}{1-\rho} + 2\frac{\|g\|_1}{(1-\rho)^2} \leqslant c_\rho \cdot \|g\|_1 \leqslant C_\rho \cdot \|g\|_p \quad \forall\, \zeta \in \partial\mathbb{D} \tag{18}$$

with $c_\rho = 3/(1-\rho)^2$ and $C_\rho = 3\pi/(1-\rho)^2$, respectively. Hence,

$$|\mathbb{S}_{\Phi_g}(z)| \leqslant C_{\rho,r} \cdot \|g\|_p, \;\; \forall\, z \in \mathbb{D}_r, \;\; r \in (0,1), \tag{19}$$

$$|\mathbb{S}_{\Phi_g}(z_1) - \mathbb{S}_{\Phi_g}(z_2)| \leqslant C^*_{\rho,r} \cdot \|g\|_p \cdot |z_1 - z_2|, \;\; \forall z_1, z_2 \in \mathbb{D}_r, r \in (0,1), \tag{20}$$

where the constants $C_{\rho,r}$ and $C^*_{\rho,r}$ depend only on the radii ρ and $r \in (0,1)$. Thus, the operator $\mathbb{S}_{\Phi_*}$ is completely continuous on compact sets in $\mathbb{D}$ again by the Arzela—Ascoli theorem. Combining it with Remark 1, we obtain the following conclusion.

Lemma 1. *Let $\lambda : \partial\mathbb{D} \to \mathbb{C}, |\lambda(\zeta)| \equiv 1$, be of countable bounded variation, and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity. Then there is a Hilbert operator H_g over $g : \mathbb{D} \to \mathbb{C}$ in $L^p(\mathbb{D})$, $p > 2$, with compact supports in $\mathbb{D}$, generating generalized analytic functions $f : \mathbb{D} \to \mathbb{C}$ with the sources g and the angular limits*

$$\lim_{z \to \zeta} \operatorname{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \qquad q.e. \text{ on } \partial\mathbb{D}, \tag{21}$$

whose restriction to sources g with $\operatorname{supp} g \subseteq \mathbb{D}_\rho$ *is completely continuous over* $\mathbb{D}_r$ *for each* ρ *and* $r \in (0,1)$.

Remark 2. Note that the nonlinear operator H_g constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $H_0 = A \cdot \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator H_g to $\mathbb{D}_r$ under each $r \in (0,1)$ is bounded at infinity in the sense that $\max\limits_{z \in \mathbb{D}_r} |H_g(z)| \leqslant M \cdot \|g\|_p$ for some $M > 0$ and all g with large enough $\|g\|_p$. Note also that, by Lemma 1 in [2], we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 on $\partial\mathbb{D}$.

2. On the Hilbert problem for semilinear equations in a unit disk. In this section, we study the solvability of the Hilbert boundary-value problem for nonlinear equations of the Vekua type $\partial_{\bar z} f(z) = h(z) q(f(z))$ in the unit disk $\mathbb{D}$. The well-known Leray—Schauder approach allows us to reduce the problem to the study of the corresponding linear equation from our paper [1] on the basis of Lemma 1 in the previous section on a completely continuous Hilbert operator H_g and Remark 2 on its boundedness at infinity.

Theorem 1. *Let $\lambda : \partial\mathbb{D} \to \mathbb{C}, |\lambda(\zeta)| \equiv 1$ be of countable bounded variation, and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity. Suppose that $h : \mathbb{D} \to \mathbb{C}$ is a function in the class $L^p(\mathbb{D})$ for $p > 2$ with compact support in $\mathbb{D}$ and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with*

$$\lim_{w \to \infty} \frac{q(w)}{w} = 0. \tag{22}$$

Then there is a function $f : \mathbb{D} \to \mathbb{C}$ in the class $C^\alpha_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar z} f(z) = h(z) \cdot q(f(z)) \qquad a.e. \text{ in } \mathbb{D} \tag{23}$$

and, in addition, f is a generalized analytic function with a source $g \in L^p(\mathbb{D})$ *and the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)}\cdot f(z)\} = \varphi(\zeta) \qquad q.e.\ on\ \partial\mathbb{D}. \tag{24}$$

Moreover, $f = H_g$, *where* H_g *is the Hilbert operator described in the last section, and the support of g is in the support of h and the upper bound of* $\| g\|_p$ *depends only on* $\| h\|_p$ *and on the function q.*

Proof. If $\| h\|_p = 0$ or $\| q\|_C = 0$, then any analytic function from Theorem 5.1 in [10] gives the desired solution of (23). Thus, we may assume that $\| h\|_p \neq 0$ and $\| q\|_C \neq 0$. Set $q_*(t) = \max_{|w|\leqslant t} |q(w)|$, $t \in \mathbb{R}^+ := [0,\infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (22),

$$\lim_{t\to\infty} \frac{q_*(t)}{t} = 0. \tag{25}$$

By Lemma 1 and Remark 2, we obtain the family of operators $F(g;\tau): L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D})$, where $L^p_h(\mathbb{D})$ consists of functions $g \in L^p(\mathbb{D})$ with supports in the support of h,

$$F(g;\tau) := \tau h\cdot q(H_g) \qquad \forall \tau\in[0,1] \tag{26}$$

which satisfies all groups of hypothesis H1-H3 of Theorem 1 in [4]. Indeed:

H1). First of all, by Lemma 1 the function $F(g;\tau) \in L^p_h(\mathbb{D})$ for all $\tau\in[0,1]$ and $g \in L^p_h(\mathbb{C})$ because the function $q(H_g)$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau\in[0,1]$ and even uniformly continuous in the parameter $\tau\in[0,1]$.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $g = F(g;\tau)$ are not bounded in $L^p_h(\mathbb{D})$, i.e., there is a sequence of functions $g_n \in L^p_h(\mathbb{D})$ with $\| g_n\|_p \to \infty$ as $n\to\infty$ such that $g_n = F(g_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$.

However, by Remark 2, we have that, for some constant $M > 0$,

$$\| g_n\|_p \leqslant \| h\|_p\, q_*(M\| g_n\|_p)$$

and, consequently,

$$\frac{q_*(M\| g_n\|_p)}{M\| g_n\|_p} \geqslant \frac{1}{M\| h\|_p} > 0 \tag{27}$$

for all large enough n. The latter is impossible by condition (25). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [4], there is a function $g \in L^p_h(D)$ with $F(g;1) = g$, and, by Lemma 1, the function $f := H_g$ gives the desired solution of (23).

Remark 3. By the construction in the above proof, the source $g:\mathbb{D}\to\mathbb{C}$ is a fixed point of the (nonlinear) integral operator $\Omega_g := h\cdot q(H_g): L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D})$, where $L^p_h(\mathbb{D})$ consists of functions g in $L^p(\mathbb{D})$ with supports in the support of h.

3. Extending results to some Jordan domains. Here, we extend the above results to Jordan domains with the so-called quasihyperbolic boundary condition, see definitions in our paper [1].

As known, such domains include, for instance, domains with quasiconformal boundaries and, in particular, domains with smooth and Lipschitz boundaries. However, quasiconformal curves can be even nowhere locally rectifiable.

Theorem 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda:\partial D\to\mathbb{C}, |\lambda(\zeta)|\equiv 1$, be in $\mathcal{CBV}(\partial D)$, and let $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to the logarithmic capacity. Suppose that $h:D\to\mathbb{C}$ is a function in the class $L^p(D)$ for $p>2$ with compact support in D and $q:\mathbb{C}\to\mathbb{C}$ is a continuous function with*

$$\lim_{w\to\infty}\frac{q(w)}{w}=0. \tag{28}$$

Then there is a function $f:D\to\mathbb{C}$ in the class $C^{\alpha}_{\mathrm{loc}}(D)$ with $\alpha=(p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar\xi} f(\xi)=h(\xi)\cdot q(f(\xi)) \qquad \text{a.e. in } \mathbb{D} \tag{29}$$

and, in addition, f is a generalized analytic function with a source $g\in L^p(D)$ and the angular limits

$$\lim_{\xi\to\omega}\mathrm{Re}\{\overline{\lambda(\omega)}\cdot f(\xi)\}=\varphi(\omega) \qquad \text{q.e. on } \partial\mathbb{D}. \tag{30}$$

Moreover, $f(\xi)=H_{\tilde g}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $H_{\tilde g}$ is the Hilbert operator described in Section 1, $\tilde g=g\circ c^{-1}$, and the support of g is in the support of h, and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$, the function q and the domain D.

Proof. Let c be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem (see, e.g., Theorem II.2.1 in [12]). Now, by the Carathéodory theorem (see, e.g., Theorem II.3.4 in [12]) c is extended to a homeomorphism $\tilde c$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Furthermore, by Corollary of Theorem 1 in [5], $c_*:=\tilde c|_{\partial D}:\partial D\to\partial\mathbb{D}$ and its inverse function are Hölder continuous. Then $\tilde\lambda:=\lambda\circ c_*^{-1}\in\mathcal{CBV}(\partial\mathbb{D})$ and $\tilde\varphi:=\varphi\circ c_*^{-1}$ is measurable with respect to the logarithmic capacity, see, e.g., Remarks 1 and 2 in [1].

Now, set $\tilde h=h\circ C\cdot\overline{C'}$, where C is the inverse conformal mapping to c, $C:=c^{-1}:\mathbb{D}\to D$. Then it is clear by the hypothesis of Theorem 2 that $\tilde h$ has a compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1, there is a function $\tilde f:\mathbb{D}\to\mathbb{C}$ in the class $C^{\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha=(p-2)/p$ and generalized first derivatives by Sobolev such that

$$\partial_{\bar z}\tilde f(z)=\tilde h(z)\cdot q(\tilde f(z)) \qquad \text{a.e. in } \mathbb{D} \tag{31}$$

and $\tilde f$ is a generalized analytic function with a source $\tilde g\in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z\to\zeta}\mathrm{Re}\{\overline{\tilde\lambda(\zeta)}\cdot\tilde f(z)\}=\tilde\varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}. \tag{32}$$

Moreover, $\tilde f=H_{\tilde g}$, where $H_{\tilde g}$ is the Hilbert operator described in Section 1, and the support of $\tilde g$ is in the support of $\tilde h$ and the upper bound of $\|\tilde g\|_p$ depends only on $\|\tilde h\|_p$ and on the function q.

Next, setting $f = \tilde{f} \circ c$, by simple calculations, see, e.g., Section 1.C in [13], we obtain that $\frac{\partial f}{\partial \bar{\xi}} = \frac{\partial \tilde{f}}{\partial \bar{z}} \circ c \cdot \overline{c'}$ and, consequently, the function $f : D \to \mathbb{C}$ is in the class $C^{\alpha}_{\text{loc}}(D)$ with $\alpha = (p-2)/2$ and generalized first derivatives by Sobolev that satisfies Eq. (29), f is a generalized analytic function with a source $g \in L^p(D)$ and, moreover, $f(\xi) = H_{\tilde{g}}(c(\xi))$, where $H_{\tilde{g}}$ is the Hilbert operator described in Section 1, $\tilde{g} = g \circ c^{-1}$, and the support of g is in the support of h and the upper bound of $\| g \|_p$ depends only on $\| h \|_p$, the function q and the domain D.

It remains to show that f has the angular limits as $\xi \to \omega \in \partial D$ and satisfies the boundary condition (30) q.e. on ∂D. Indeed, by the Lindelöf theorem, see, e.g., Theorem II.C.2 in [14], if ∂D has a tangent at a point ω, then $\arg[c_*(\omega) - c(\xi)] - \arg[\omega - \xi] \to \text{const}$ as $\xi \to \omega$. In other words, the images under the conformal mapping c of sectors in D with a vertex at $\omega \in \partial D$ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $\zeta = c_*(\omega) \in \partial\mathbb{D}$. Consequently, nontangential paths in D are transformed under c into nontangential paths in $\mathbb{D}$ and inversely q.e. on ∂D and $\partial\mathbb{D}$, respectively, because ∂D has a tangent q.e. and c_* and c_*^{-1} keep sets of logarithmic capacity zero.

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В.Я. Гутлянський [1], https://orcid.org/0000-0002-8691-4617
О.В. Нєсмєлова [1, 2], https://orcid.org/0000-0003-2542-5980
В.І. Рязанов [1, 3], https://orcid.org/0000-0002-4503-4939
А.С. Єфімушкін[1]

[1] Інститут прикладної математики і механіки НАН України, Слов'янськ
[2] Донбаський державний педагогічний університет, Слов'янськ
[3] Черкаський національний університет ім. Богдана Хмельницького
E-mail: vgutlyanskii@gmail.com, star-o@ukr.net, vl.ryazanov1@gmail.com, a.yefimushkin@gmail.com

ЗАДАЧА ГІЛЬБЕРТА З ВИМІРНИМИ ДАНИМИ ДЛЯ НАПІВЛІНІЙНИХ РІВНЯНЬ ТИПУ ВЕКУА

Вивчення задачі Діріхле з довільними вимірними даними для гармонічних функцій в одиничному колі $\mathbb{D}$ сходить до відомої дисертації Лузіна. Пізніше Векуа були досліджені граничні задачі тільки з неперервними за Гельдером даними для узагальнених аналітичних функцій, тобто неперервних комплекснозначних функцій $f(z)$ комплексної змінної $z = x + iy$ з узагальненими першими частинними похідними за Соболєвим, які задовольняють рівняння виду $\partial_{\bar{z}} f + af + b\overline{f} = c$, де передбачалося, що комплекснозначні функції a, b і c належать класу L^p, $p > 2$, у досить гладких областях D в $\mathbb{C}$.

Дана робота містить теореми існування розв'язків граничної задачі Гільберта з довільними вимірними даними для відповідних нелінійних рівнянь типу Векуа $\partial_{\bar{z}} f(z) = h(z)q(f(z))$. Знайдені розв'язки не є класичними, оскільки наш підхід базується на інтерпретації граничних значень у сенсі кутових (вздовж недотичних шляхів) границь, що є традиційним інструментом геометричної теорії функцій, але не рівнянь у частинних похідних. Одержані результати можуть бути застосовані до встановлення теорем існування для граничної задачі Пуанкаре і, зокрема, для задачі Неймана для нелінійних рівнянь Пуасона виду $\Delta U(z) = H(z)Q(U(z))$ з довільними вимірними даними відносно логарифмічної ємності. Таким чином, вони можуть буть застосовані також до напівлінійних рівнянь математичної фізики під час моделювання різних фізичних процесів, таких як дифузія з абсорбцією, стани плазми, стаціонарне горіння і т. д. в анізотропних і неоднорідних середовищах. Останнє буде змістом наших подальших статей.

Ключові слова: *гранична задача Гільберта, вимірні граничні дані, логарифмічна ємність, напівлінійні рівняння типу Векуа, нелінійні джерела, кутові границі, недотичні шляхи.*