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Poincaré problem with measurable data for semilinear Poisson equation in the plane

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We study the Poincaré boundary-value problem with measurable in terms of the logarithmic capacity boundary data for semilinear Poisson equations defined either in the unit disk or in Jordan domains with quasihyperbolic boundary condition. The solvability theorems as well as their applications to some semilinear equations, modelling diffusion with absorption, plasma states and stationary burning, are given.

Keywords: *Poincaré and Neumann boundary-value problems, measurable boundary data, logarithmic capacity, semilinear equations of the Poisson type, nonlinear sources, angular limits, nontangent paths.*

1. On completely continuous Poincaré operators. In Section 7 of [1], we considered the Poincaré boundary-value problem in terms of directional derivatives for the Poisson equations

$$\Delta U(z) = G(z) \tag{1}$$

and, as a partial case, the corresponding Neumann problem with arbitrary measurable boundary data with respect to logarithmic capacity. Here G stands for the real-valued function defined in Jordan domains $D \subset \mathbb{C}$ and it belongs to the spases $L^p(D)$ with $p > 2$. Recall that a continuous solution $U \in W^{2,p}_{\text{loc}}(D)$ of (1) was called in [1] as a *generalized harmonic function with the source* G. Such a solution, by the Sobolev embedding theorem, belongs to the class C^1, see Theorem I.10.2 in [2].

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From now on, $\frac{\partial u}{\partial \nu}(\xi)$ denotes the derivative of u at the point $\xi \in D$ in the direction $\nu \in \mathbb{C}$, $|\nu|=1$, i. e.,

$$\frac{\partial u}{\partial \nu}(\xi) := \lim_{t \to 0} \frac{u(\xi + t \cdot \nu) - u(\xi)}{t}. \tag{2}$$

Recall that the Neumann boundary value problem is a special case of the Poincaré problem on the directional derivatives with the unit interior normal $n = n(\omega)$ to ∂D at the point ω as $\nu(\omega)$, see Corollary 1 below.

By Theorem 5 in [1], for each $\nu : \partial \mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, in $\mathcal{CBV}(\partial \mathbb{D})$, and $\varphi : \partial \mathbb{D} \to \mathbb{R}$, that is measurable with respect to logarithmic capacity, and $G : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, there is a generalized harmonic function $U : \mathbb{D} \to \mathbb{R}$ with the source G in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ that have the angular limits

$$\lim_{z \to \zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \qquad \text{quasi everywhere on } \partial D. \tag{3}$$

Furthermore, the space of all such functions U has the infinite dimension.

By its proof, assuming that G have compact supports in $\mathbb{D}$, one of such functions U can be presented as the sum of the logarithmic (Newtonian) potential

$$N_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w| G(w)\, dm(w) \tag{4}$$

and the harmonic function

$$\gamma(z) := \operatorname{Re} \int_0^z \{\mathcal{H}_{G/2}(\xi) - T_{G/2}(\xi)\} d\zeta \tag{5}$$

with the known Pompeiu integral operator

$$T_g(z) := \frac{1}{\pi} \int_{\mathbb{C}} g(w) \frac{dm(w)}{z - w} \tag{6}$$

and the Hilbert operator $\mathcal{H}_g$ from Section 1 in [3], generating generalized analytic functions $f : \mathbb{D} \to \mathbb{C}$ with the sources g and the angular limits

$$\lim_{z \to \zeta} \operatorname{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \qquad q.e. on\ \partial \mathbb{D}, \tag{7}$$

whose restriction to sources g with $\operatorname{supp} g \subseteq \mathbb{D}_\rho := \{z \in \mathbb{C} : |z| \leqslant \rho\}$ is completely continuous over $\mathbb{D}_r := \{z \in \mathbb{C} : |z| \leqslant r\}$ for each ρ and $r \in (0,1)$.

Denoting by the given correspondence between G and the generalized harmonic functions with the sources G and the Poincaré boundary condition (3), we see that $\mathcal{P}_G$ is a completely continuous operator over each disk $\mathbb{D}_r$, $r \in (0,1)$, because the operators $\mathcal{H}_{G/2}$ and $T_{G/2}$ are so and, in addition, the indefinite integral, as well as the operator of taking Re is bounded and linear. Thus, by Lemma 1 and Remark 2 in [3], we come to the following statements.

Lemma 1. *Let* $\nu:\partial\mathbb{D}\to\mathbb{C},:|\nu(\zeta)|\equiv 1$, *be of countable bounded variation and* $\varphi:\partial\mathbb{D}\to\mathbb{R}$ *be measurable with respect to logarithmic capacity. Then there is a Poincaré operator* $\mathcal{P}_G$ *over the sources* $G:\mathbb{D}\to\mathbb{R}$ *in* $L^p(\mathbb{D})$, $p>2$, *with compact supports in* $\mathbb{D}$, *which generates generalized harmonic functions* $U:\mathbb{D}\to\mathbb{R}$ *with the sources G and the angular limits* (3), *whose restriction to sources G with* $\mathrm{supp}G\subseteq\mathbb{D}_\rho$ *is completely continuous over* $\mathbb{D}_r$ *for each* ρ *and* $r\in(0,1)$.

Remark 1. Furthermore, we may assume that the restriction of the operator $\mathcal{P}_G$ to $\mathbb{D}_r$ under each $r\in(0,1)$ is bounded at infinity in the sense that $\max_{z\in\mathbb{D}_r}|\mathcal{P}_G(z)|\leqslant M\cdot\|G\|_p$ for some $M>0$ and all G with large enough $\|G\|_p$.

2. On the Poincaré problem in the unit disk. In this section, we study the solvability of the Poincaré boundary-value problem on directional derivatives in the unit circle for semilinear Poisson equations of the form $\Delta U(z)=H(z)\cdot Q(U(z))$ in the unit disk $\mathbb{D}$.

Theorem 1. *Let* $\nu:\partial\mathbb{D}\to\mathbb{C},:|\nu(\zeta)|\equiv 1$, *be of countable bounded variation and* $\varphi:\partial\mathbb{D}\to\mathbb{R}$ *be measurable with respect to logarithmic capacity. Suppose that* $H:\mathbb{D}\to\mathbb{R}$ *is a function in the class* $L^p(\mathbb{D})$ *for* $p>2$ *with compact support in* $\mathbb{D}$ *and* $Q:\mathbb{R}\to\mathbb{R}$ *is a continuous function with*

$$\lim_{t\to\infty}\frac{Q(t)}{t}=0. \tag{8}$$

Then there is a function $U:\mathbb{D}\to\mathbb{R}$ *in the class* $W^{2,p}_{\mathrm{loc}}(\mathbb{D})\cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ *with* $\alpha=(p-2)/p$ *such that*

$$\Delta U(z)=H(z)\cdot Q(U(z))\qquad a.e.\ in\ D \tag{9}$$

and, in addition, U is a generalized harmonic function with a source $G\in L^p(\mathbb{D})$ *and the angular limits*

$$\lim_{z\to\zeta}\frac{\partial U}{\partial\nu}(z)=\varphi(\zeta)\qquad q.e.\ on\ \partial D\,. \tag{10}$$

Moreover, $U=\mathcal{P}_G$, *where* $\mathcal{P}_G$ *is the Poincaré operator described in the last section, the support of G is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$ *and on the function Q.*

Proof. If $\|H\|_p=0$ or $\|Q\|_C=0$, then any harmonic function from Theorem 7.2 in [4] gives a desired solution of (9). Thus, we may assume that $\|H\|_p\neq 0$ and $\|Q\|_C\neq 0$. Set $Q_*(t)=\max_{|\tau|\leqslant t}|Q(\tau)|$, $t\in\mathbb{R}^+:=[0,\infty)$. Then the function $Q_*:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, nondecreasing, and, by (8),

$$\lim_{t\to\infty}\frac{Q_*(t)}{t}=0. \tag{11}$$

By Lemma 1 and Remark 1, we obtain the family of operators $F(G;\tau):L^p_H(\mathbb{D})\to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions $G\in L^p(\mathbb{D})$ with supports in the support of H,

$$F(G;\tau):=\tau H\cdot Q(\mathcal{P}_G)\qquad\forall\,\tau\in 0,1] \tag{12}$$

which satisfies all groups of hypothesis H1–H3 of Theorem 1 in [5]. Indeed:

H1) First of all, by Lemma 1, the function $F(G;\tau)\in L^p_H(\mathbb{D})$ for all $\tau\in 0,1]$ and $G\in L^p_H(\mathbb{C})$ because the function $Q(\mathcal{P}_G)$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau \in 0,1]$ and even uniformly continuous with respect to the parameter $\tau \in 0,1]$.

H2) The index of the operator $F(\cdot;0)$ is obviously equal to 1.

H3) Assume that the solutions of the equations $G = F(G;\tau)$ are not bounded in $L_H^p(\mathbb{D})$, i. e., there is a sequence of functions $G_n \in L_H^p(\mathbb{D})$ with $\|G_n\|_p \to \infty$ as $n \to \infty$ such that $G_n = F(G_n;\tau_n)$ for some $\tau_n \in 0,1]$, $n = 1, 2, \ldots$. However, then, by Remark 1, we have that for some constant $M > 0$,

$$\|G_n\|_p \leqslant \|H\|_p \, Q_*(M\|G_n\|_p)$$

and, consequently,

$$\frac{Q_*(M\|G_n\|_p)}{M\|G_n\|_p} \geqslant \frac{1}{M\|H\|_p} > 0 \tag{13}$$

for all large enough *n*. The latter is impossible by condition (11). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [5], there is a function $G \in L_H^p(D)$ with $F(G;1) = G$, and by Lemma 1, the function $U := \mathcal{P}_G$ gives a desired solution of (9).

Remark 2. By the construction in the above proof, the source $G : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\Omega_G := H \cdot Q(\mathcal{P}_G) : L_H^p(\mathbb{D}) \to L_H^p(\mathbb{D})$, where ( ) consists of functions *G* in $L^p(\mathbb{D})$ with supports in the support of *H*.

3. On the Poincaré problem in Jordan domains. Now we extend the above results to Jordan domains with the so-called quasihyperbolic boundary condition, see the definition e. g. in [4]. Recall here only that such domains include, for instance, domains with quasiconformal boundaries and, in particular, domains with smooth and Lipschitz boundaries. However, the mentioned quasiconformal curves can be even nowhere locally rectifiable.

Theorem 2. *Let D be a Jordan domain with a quasihyperbolic boundary condition, ∂D have a tangent* q. e., $\nu : \partial D \to \mathbb{C}, : |\nu| \equiv 1$, *be in $\mathcal{CBV}(\partial D)$, and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for p > 2 with compact support in D, and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with*

$$\lim_{t \to \infty} \frac{Q(t)}{t} = 0. \tag{14}$$

Then there is a function $U : D \to \mathbb{R}$ in the class $W_{\mathrm{loc}}^{2,p}(D) \cap C_{\mathrm{loc}}^{1,\alpha}(D)$ with $\alpha = (p-2)/p$ such that

$$\Delta U(\xi) = H(\xi) \cdot Q(U(\xi)) \qquad a.e.\ in\ D \tag{15}$$

and, in addition, U is a generalized harmonic function with a source $G \in L^p(D)$ and with the angular limits

$$\lim_{\xi \to \omega} \frac{\partial U}{\partial \nu}(\xi) = \varphi(\omega) \qquad q.e.\ on\ \partial D. \tag{16}$$

Moreover, $U(\xi) = \mathcal{P}_{\tilde{G}}(c(\xi))$, where c is a conformal mapping of D onto $\mathbb{D}$, $\mathcal{P}_{\tilde{G}}$ is the Poincaré operator described in Section 1, $\tilde{G} = G \circ c^{-1}$, the support of G is in the support of H, and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

Proof. Arguing similarly to the first item in the proof of Theorem 2 in [3], we see that $\tilde{\nu} := \nu \circ c_*^{-1} \in \mathcal{CBV}(\partial\mathbb{D})$ and $\tilde{\varphi} := \varphi \circ c_*^{-1}$ is measurable with respect to logarithmic capacity, where $c_* := \tilde{c}|_{\partial D} : \partial D \to \partial\mathbb{D}$ is the restriction to the boundary of the homeomorphic extension of c to $\overline{D}$ onto $\overline{\mathbb{D}}$.

Now, set $\tilde{H} = |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypotheses of Theorem 2 that $\tilde{H}$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1, there is a function $\tilde{U} : \mathbb{D} \to \mathbb{R}$ in the class $W^{1,p}_{\text{loc}}(\mathbb{D}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ such that

$$\Delta\tilde{U}(z) = \tilde{H}(z) \cdot Q(\tilde{U}(z)) \qquad a.e.\ in\ \mathbb{D} \tag{17}$$

and $\tilde{U}$ is a generalized analytic function with a source $\tilde{G} \in L^p(\mathbb{D})$ and the angular limits

$$\lim_{z\to\zeta} \frac{\partial\tilde{U}}{\partial\tilde{\nu}}(z) = \tilde{\varphi}(\zeta) \qquad q.e.\ on\ \partial\mathbb{D}. \tag{18}$$

Moreover, $\tilde{U} = \mathcal{P}_{\tilde{G}}$, where $\mathcal{P}_{\tilde{G}}$ is the Poincaré operator described in Section 1, the support of $\tilde{G}$ is in the support of $\tilde{H}$, and the upper bound of $\|\tilde{G}\|_p$ depends only on $\|\tilde{H}\|_p$ and the function Q.

Next, setting $U = \tilde{U} \circ c$ and by simple calculations, see e. g. Section 1.C in [6], we obtain that $\Delta U = |c'|^2 \cdot \Delta\tilde{U} \circ c$ and, consequently, the function $U : D \to \mathbb{C}$ is in the class $W^{1,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ with $\alpha = (p-2)/p$ that satisfies equation (15), U is a generalized harmonic function with a source $G \in L^p(D)$. In addition, $U(\xi) = \mathcal{P}_{\tilde{G}}(c(\xi))$, where $\mathcal{P}_{\tilde{G}}$ is the Poincaré operator from Section 1, $\tilde{G} = G \circ c^{-1}$, the support of G is in the support of H, and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q, and the domain D.

Finally, arguing similarly to the last item in the proof of Theorem 2 in [3], we show that (18) implies (16).

Remark 3. By the construction in the above proof, the source $G = \tilde{G} \circ c$, where c is a conformal mapping of D onto $\mathbb{D}$ and $\tilde{G} : \mathbb{D} \to \mathbb{R}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_{G_*} := \tilde{H} \cdot Q(\mathcal{P}_{G_*}) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$. Here $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions G_* in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H} := |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$.

We are able to say more in Theorem 2 for the case of $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, $n(\zeta)\overline{\nu(\zeta)}$ is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν onto n. In view of (16), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and parallel to the vector ν because along this line,

$$U(z) = U(z_0) - \int_0^1 \frac{\partial U}{\partial\nu}(z_0 + \tau(z - z_0))\,d\tau. \tag{19}$$

Thus, at each point with condition (16), there is the directional derivative

$$\frac{\partial U}{\partial\nu}(\zeta) := \lim_{t\to 0} \frac{U(\zeta + t\cdot\nu) - U(\zeta)}{t} = \varphi(\zeta). \tag{20}$$

In particular, in the case of the Neumann problem, $\mathrm{Re}\, n(\zeta)\overline{\nu(\zeta)} \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have by Theorem 2 and Remark 3 the following significant result.

Corollary 1. *Let D be a Jordan domain in* $\mathbb{C}$ *with the quasihyperbolic boundary condition, the unit inner normal* $n(\zeta)$, $\zeta \in \partial D$, *belong to the class* $\mathcal{CBV}(\partial D)$, *and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ *is in* $L^p(D)$, $p > 2$, *with compact support in D. Then one can find a generalized harmonic function* $U : D \to \mathbb{R}$ *with a source* $G \in L^p(D)$ *satisfying equation* (15) *such that, q.e. on* ∂D, *there exist:*

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) := \lim_{z\to\zeta} U(z);$$

2) the normal derivative

$$\frac{\partial U}{\partial n}(\zeta) := \lim_{t\to 0} \frac{U(\zeta + t\cdot n(\zeta)) - U(\zeta)}{t} = \varphi(\zeta);$$

3) the angular limit

$$\lim_{z\to\zeta} \frac{\partial U}{\partial n}(z) = \frac{\partial U}{\partial n}(\zeta).$$

4. The Poincaré problem in physical applications. The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [7, p. 4] and, in detail, in [8]. A nonlinear system is obtained for the density *U* and the temperature *T* of the reactant. Upon eliminating *T*, the system can be reduced to equations of type (15), $\Delta U = \sigma \cdot Q(U)$ with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^{\beta}$, $\beta > 0$. It turns out that the density of the reactant *U* may be zero in a subdomain called the dead core. A particularization of results in Chapter 1 of [7] shows that a dead core may exist just if and only if $\beta \in (0,1)$ and σ is large enough, see also the corresponding examples in [9]. In this connection, the following statements may be of independent interest.

Corollary 2. *Let D be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent q. e.,* $\nu : \partial D \to \mathbb{C}, : |\nu| \equiv 1$, *be in* $\mathcal{CBV}(\partial D)$, *and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to logarithmic capacity.*

Suppose that $H : D \to \mathbb{R}$ *is a function in the class* $L^p(D)$ *for p > 2 with compact support in D. Then there is a solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\mathrm{loc}}(D) \cap C^{1,\alpha}_{\mathrm{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation*

$$\Delta U(\xi) = H(\xi)\cdot U^{\beta}(\xi), \quad 0 < \beta < 1, \qquad a.e.\ in\ D \tag{21}$$

satisfying the Poincaré boundary condition on directional derivatives

$$\lim_{\xi\to\omega} \frac{\partial U}{\partial \nu}(\xi) = \varphi(\omega) \qquad q.e.\ on\ \partial D \tag{22}$$

in the sense of the angular limits. Moreover, U is a generalized harmonic function with a source $G \in L^p(D)$*, whose support is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$*, the function Q, and the domain D.*

Corollary 3. *In particular, in the case of Neumann problem, i. e., if* $\nu(\zeta)$ *is the unit interior normal* $n(\zeta)$ *to* ∂D *at the point* ζ*, one can find a solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation* (21) *that satisfies the conclusions 1-3 of Corollary* 1 *q. e. on* ∂D .

Note that certain mathematical models of the thermal evolution of heated plasma also lead to nonlinear equations of type (15). Indeed, it is known that some of them have the form $\Delta\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if / as, for instance, $\psi(u) = |u|^{q-1} u$ under $0 < q < 1$, see e. g. [7]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \text{sign} u$, we have that $u = |U|^Q \cdot \text{sign} U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \text{sign} u$, we come to the equation $\Delta U = |U|^q \cdot \text{sign} U = \psi(U)$.

Corollary 4. *Let D be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent* q. e., $\nu : \partial D \to \mathbb{C}, : |\nu| \equiv 1$*, be in* $\mathcal{CBV}(\partial D)$*, and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to logarithmic capacity.*

Suppose also that $H : D \to \mathbb{R}$ *is a function in the class* $L^p(D)$ *for p > 2 with compact support in D. Then there is a solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation*

$$\Delta U(\xi) = H(\xi) \cdot |U(\xi)|^{\beta-1} U(\xi), \ 0 < \beta < 1, \qquad a.e. \ in \ D \tag{23}$$

satisfying *the Poincaré boundary condition on directional derivatives* (22). *Moreover, U is a generalized harmonic function with a source* $G \in L^p(D)$ *whose support is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$*, the function Q, and the domain D.*

Corollary 5. *In particular, in the case of Neumann problem, i. e., if* $\nu(\zeta)$ *is the unit interior normal* $n(\zeta)$ *to* ∂D *at the point* ζ*, one can find a solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation* (23) *that satisfies the conclusions 1-3 of Corollary* 1 *q. e. on* ∂D *. Moreover, U is a generalized harmonic function with a source* $G \in L^p(D)$ *whose support is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$*, the function Q, and the domain D.*

Finally, we recall that in the combustion theory, see e. g. [10, 11] and the references therein, we met the nonlinear sources of the exponential type. Note that the corresponding equation of type (15) appears here after the replacement of the function *u* by −*u*, with the function $Q(u) = e^{-u}$ that is bounded at all.

Corollary 6. *Let D be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent* q. e., $\nu : \partial D \to \mathbb{C}, : |\nu| \equiv 1$*, be in* $\mathcal{CBV}(\partial D)$*, and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to logarithmic capacity.*

Suppose also that $H : D \to \mathbb{R}$ *is a function in the class* $L^p(D)$ *for p > 2 with compact support in D. Then there is a solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation*

$$\Delta U(\xi) = H(\xi) \cdot e^{U(\xi)}, \qquad a.e. \ in \ D \tag{24}$$

satisfying the Poincaré boundary condition on directional derivatives (22). *Moreover, U is a generalized harmonic function with a source* $G \in L^p(D)$ *whose support is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$, *the function Q, and the domain D.*

Corollary 7. *In particular, in the case of Neumann problem, i. e., if* $\nu(\zeta)$ *is the unit interior normal* $n(\zeta)$ *to* ∂D *at the point* ζ, *one can find a solution* $U: D \to \mathbb{R}$ *in the class* $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ *with* $\alpha = (p-2)/p$ *of the semilinear Poisson equation* (24) *that satisfies the conclusions 1-3 of Corollary* 1 *q. e. on* ∂D. *Moreover, U is a generalized harmonic function with a source* $G \in L^p(D)$ *whose support is in the support of H, and the upper bound of* $\|G\|_p$ *depends only on* $\|H\|_p$, *the function Q, and the domain D.*

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ЗАДАЧА ПУАНКАРЕ З ВИМІРНИМИ ДАНИМИ ДЛЯ НАПІВЛІНІЙНИХ РІВНЯНЬ ПУАССОНА НА ПЛОЩИНІ

Крайова задача Гільберта належить до найважливіших з огляду на її численні застосування, зокрема, до крайових задач Діріхле, Пуанкаре та Неймана в гідромеханіці. Перший підхід до її розв'язання був запропонований самим Гільбертом і заснований на теорії сингулярних інтегральних рівнянь. На цьому шляху доведено існування її розв'язків для неперервних за Гельдером граничних даних. Лузін уперше встановив існування розв'язків задачі Діріхле при довільних вимірних даних для гармонічних функцій в одиничному крузі в термінах кутових (недотичних) границь м. в. на одиничному колі. Раніше нами були сформульовані теореми існування розв'язків крайової задачі Гільберта при довільних вимірних даних для узагальнених гармонічних функцій з джерелами. Знайдені розв'язки не були класичними, оскільки наш підхід ґрунтувався на інтерпретації граничних значень у сенсі кутових (недотичних) границь, що стало традиційним інструментом геометричної теорії функцій, але не PDE.

Представлена стаття містить аналогічні теореми існування розв'язків задачі Пуанкаре про похідні за напрямками на межі і, зокрема, задачі Неймана при довільних граничних даних вимірних відносно логарифмічної ємності уздовж недотичних шляхів для напівлінійних рівнянь Пуассона. Наведено застосування цих результатів до деяких напівлінійних рівнянь математичної фізики, що моделюють різні фізичні процеси, такі як дифузія з абсорбцією, процес стаціонарного горіння та стани плазми.

Ключові слова: *крайові задачі Пуанкаре і Неймана, вимірні граничні дані, лограифмічна ємність, напівлінійні рівняння типу Пуассона, нелінійні джерела, кутові границі, недотичні шляхи.*