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On some commutative invariants of modules over minimax nilpotent groups

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In the paper we introduce a finite system of invariants for modules over minimax nilpotent groups which consists of classes of equivalent prime ideals of the group algebra of an Abelian minimax group. In particuly, introduced system of invariants allows to study the structure of a minimax nilpotent group N acted by a group of operators G such that N has injective modules which are stabilized by the group of operators G.

Keywords: *nilpotent groups, minimax groups, group rings.*

A group G is said to have finite (Prufer) rank if there is a positive integer m such that any finitely generated subgroup of G may be generated by m elements; the smallest m with this property is the rank $r(G)$ of G. A group G is said to be of finite torsion-free rank if it has a finite series each of whose factor is either infinite cyclic or locally finite; the number $r_0(G)$ of infinite cyclic factors in such a series is the torsion-free rank of G. If the group G has a finite series each of whose factor is either cyclic or quasi-cyclic then G is said to be minimax. If in such a series all factors are cyclic then the group G is said to be polycyclic.

Let H be a subgroup of a group G, the subgroup H is said to be dense in G if for any $g \in G$ there is an integer $n \in \mathbb{N}$ such that $g^n \in H$. If $g^n \in G \setminus H$ for any $n \in \mathbb{N}$ and any $g \in G \setminus H$ then the subgroup H is said to be isolated in G. If the group G is locally nilpotent then the isolator $\mathrm{is}_G(H) = \{g \in G \mid g^n \in H \text{ for some } n \in \mathbb{N}\}$ of H in G is a subgroup of G and if H is a normal subgroup then so is $\mathrm{is}_G(H)$.

We need some notations introduced by Wilson in [1, section 3.8] and based on the results of [2, 3]. Let A be a torsion-free abelian group of finite rank and let k be a field. If I and J are ideals of kA then we write $I \approx J$ if $I \cap kB = J \cap kB$ for some finitely generated dense subgroup $B \leqslant A$. Then $\approx$ is an equivalence relation on the set of all ideals of kA and we denote by $[J]$ the class of equivalence containing an ideal J of kA. If the ideal J is proper we will say that the class $[J]$ is

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proper. If a group G acts on A then we obtain an action of G on the set of equivalent prime ideals of kA which is given by $[J]^g = [J^g]$.

If B is a dense subgroup of A and P is a prime ideal of kB then, as kA is an integer domain over kB, it follows from [4, Chap. V, § 2 Theorem 1] that there is a prime ideal Q of kA such that $Q \cap kB = P$ and we put $[P]_{kA} = [Q]$. If μ is a set of prime ideals of kB then we put $[\mu]_{kA} = \{[P]_{kA} | P \in \mu\}$.

Let S be a commutative ring and let J be an ideal of S. Then $\mu_S(J)$ denotes the set of all prime ideals of S, which are minimal over J. If the ring S is Noetherian then it follows from [4, Chap. IV, § 1.4, Theorem 2] that the set $\mu_S(J)$ is finite.

Let J be an ideal of kA. We say that a finitely generated dense subgroup C of A is J-local if for any subgroup X of finite index in C the mapping $\mu_{kC}(J \cap kC) \to \mu_{kX}(J \cap kX)$ given by $P \mapsto P \cap kX$ is bijective. By [5, Theorem 3.7], the group ring kC is a Noetherian and hence the set $\mu_{kC}(J \cap kC)$ is finite.

Proposition 1. *Let A be a torsion-free abelian group of finite rank and let k be a field. Let B be a dense subgroup of A and let J be an ideal of kB. Then:*

(i) B contains a J-local subgroup;

(ii) for any J-local subgroup $C \leqslant B$ *we have* $|[\mu_{kC}(J \cap kC)_{kA}]| = |\mu_{kC}(J \cap kC)| < \infty$;

(iii) for any J-local subgroups C, D $\leqslant$ *B we have* $[\mu_{kC}(J \cap kC)]_{kA} = [\mu_{kD}(J \cap kD)]_{kA}$.

Let G be a group and let K be a normal subgroup of G. Let R be a ring and let I be a G-invariant ideal of the group ring RK then $I^\dagger = G \cap (I+1)$ is a G-invariant subgroup of G.

We say that the ideal I is G-large if $R/(R \cap I) = k$ is a field, $|K/I^\dagger| < \infty$ and $I = (RF \cap I)RK$, where F is a G-invariant subgroup of K such that $I^\dagger \leqslant F$ and the quotient group $F/I^\dagger$ is abelian. If R is a field then, certainly, $R = k$.

Let N be a normal subgroups of G such that $K \leqslant N \leqslant G$ and the quotient group N/K is torsion-free abelian of finite rank. Let k be a field and let I be a G-large ideal of kK. Then, as the quotient group $K/I^\dagger$ is finite, the derived subgroup of the quotient group $N/I^\dagger$ is finite and it is not difficult to show that $N/I^\dagger$ has a characteristic central subgroup A of finite index. Since the quotient group N/K is torsion-free abelian, the subgroup A may be taken torsion-free. Let W be a kN-module then W/WI may be considered as kA-module. Since the group algebra kA is commutative, we can apply methods of commutative algebra for studying the module W. This approach was introduced by Brookes in [6] for the case where the group G is polycyclic-by-finite.

Let L be a dense subgroups of N such that $K \leqslant L$ and the quotient groups L/K is finitely generated. Then $B = A \cap L/I^\dagger$ is a dense central finitely generated torsion-free subgroup of A. By [5, Theorem 3.7], the group ring kB is a Noetherian. Let V be a finitely generated RL-module then V/VI is a finitely generated kB -module and hence V/VI is a Noetherian kB-module.

By [4, Chap. IV, § 1.4, Theorem 2], for any commutative Noetherian ring S and any Noetherian S-module M the set $\mu_S(M) = \mu_S(\mathrm{Ann}_S M)$ of prime ideals of S, which are minimal over $\mathrm{Ann}_S M$, is finite. Therefore, we can define a finite set $\mu_{kB}(V/VI) = \mu_{kB}(\mathrm{Ann}_{kB}(V/VI))$ of prime ideals of kB.

Let W be a kN-module which is kK-torsion-free and let akL be a cyclic kL-module generated by an element $0 \neq a \in W$. Let a_L be the image of the element a in the quotient module $akL/akLI$. By Proposition 1(i), there is a finitely generated dense $\mathrm{Ann}_{kA}(a_L)$-local subgroup $A_L \leqslant A \cap L/I^\dagger = B$.

As A_L is a finitely generated dense subgroup of finite index in $A \cap L/I^\dagger$, we can conclude that A_L is a dense finitely generated torsion-free subgroup of A and $akL/akLI$ is a finitely generated kA_L-module. Then it follows from [5, Theorem 3.7] that the domain kA_L is Noetherian and $akL/akLI$ is a Noetherian kA_L-module. Thus, the finite set $\mu_{kA_L}(akL/akLI)$ is well defined.

As $akL/akLI = a_L k(L/I^{\dagger})$ and A_L is a central subgroup of finite index in $L/I^{\dagger}$, we see that $\mathrm{Ann}_{kA_L}(akL/aklI) = \mathrm{Ann}_{kA_L}(a_L)$ and hence

$$\mu_{kA_L}(akL/akLI) = \mu_{kA_L}(\mathrm{Ann}_{kA_L}(akL/aklI)) = \mu_{kA_L}(\mathrm{Ann}_{kA_L}(a_L)). \qquad (1)$$

Thus, the set $\mu_{kA_L}(akL/akLI)$ is defined for any $0 \neq a \in W$ and we put $[\mu_{kA_L}(akL/akLI)]_{kA} = \{[P]_{kA} \mid P \in \mu_{kA_L}(akL/akLI)\}$. It follows from (1) that

$$[\mu_{kA_L}(akL/akLI)]_{kA} = [\mu_{kA_L}(\mathrm{Ann}_{kA_L}(bkL/bkLI))]_{kA} = [\mu_{kA_L}(\mathrm{Ann}_{kA_L}(a_L))]_{kA}. \qquad (2)$$

Then, by (2) and according to Proposition 1 (*ii*), (*iii*), the set $[\mu_{kA_L}(akL/akLI)]_{kA} = \{[P]_{kA} \mid P \in \mu_{kA_L}(akL/akLI)\}$ is finite and does not depend on the choice of the $\mathrm{Ann}_{kA}(a_L)$-local subgroup A_L. Note that everywhere below in the definition of the set $[\mu_{kB}(akL/akLI)]_{kA} = \{[P]_{kA} \mid P \in \mu_{kA_L}(akL/akLI)\} = \{[P]_{kA} \mid P \in \mu_{kA_L}(\mathrm{Ann}_{kA_L}(a_L))\}$ we assume that the subgroup A_L is $\mathrm{Ann}_{kA}(a_L)$-local.

Proposition 2. *Let N be a minimax torsion-free nilpotent group and let K be a normal subgroup of N such that the quotient group N/K is torsion-free abelian. Let k be a field of characteristic zero and let W be a kN-module which is kK-torsion-free. Let I be an N-large ideal of kK and A be a torsion-free characteristic central subgroup of finite index in* $N/I^{\dagger}$. *Then there exists a cyclic kN-submodule* $0 \neq V \leqslant W$ *such that* $[\mu_{kA_L}(akL/akLI)]_{kA} = [\mu_{kA_M}(bkM/bkMI)]_{kA}$ *for any elements* $0 \neq a,b \leqslant V$ *and any dense subgroups* $L,M \leqslant N$ *such that* $K \leqslant L \cap M$ *and the quotient groups L/K and M/K are finitely generated*.

The above Proposition shows that we can define the set $M_{kA}(V/VI) = [\mu_{kA_L}(akL/akLI)]_{kA}$ which does not depends on the choice of an element $0 \neq a \in V$ and a dense subgroup $L \leqslant N$ such that $K \leqslant L$ and the quotient groups L/K is finitely generated. So, the set $M_{kA}(V/VI)$ may be considered as a finite set of invariants of the module V. By the definition, the set $M_{kA}(V/VI) = [\mu_{kA_L}(akM/akMI)]_{kA}$ consists of equivalence classes which are defined by prime ideals of the commutative group algebra kA. The following Theorem considers some important properties of the set $M_{kA}(V/VI)$.

Theorem 1. *Let G be a soluble group of finite torsion-free rank, let N be a minimax nilpotent torsion-free normal subgroup of G and let K be a G-invariant subgroup of N such that the quotient group N/K is torsion-free abelian. Let k be a field of characteristic zero and let W be a kN-module which is kN-torsion and kK-torsion-free. Let X be an isolated subgroup of N such that* $K \leqslant X$ *and the module W is kX-torsion-free. Then there are a cyclic kN-submodule* $0 \neq V \leqslant W$, *a G-large ideal I of kK and a central G-invariant subgroup A of finite index in* $N/I^{\dagger}$ *such that:*

(i) $M_{kA}(V/VI) = M_{kA}(akN/akNI)$ *for any element* $0 \neq a \in V$;

(ii) the kA-module V/VI is kA-torsion but not *kB*-torsion, where $B = A \cap (X/I^{\dagger})$;

(iii) for any $g \in G$ *we have* $M_{kA}(Vg/VgI) = M_{kA}(V/VI)^g = \{[P]^g = [P^g] \| [P] \in M_{kA}(V/VI)\}$.

Let A be an abelian torsion-free group acted by a group G, we consider A as an additive group. Let $\overline{A} = A \otimes_{\mathbb{Z}} \mathbb{Q}$, we denote by $\mathrm{Soc}_G(\overline{A})$ the socle of the $\mathbb{Q}G$-module $\overline{A}$ and put

$\operatorname{Soc}_G(A) = \operatorname{Soc}_G(\bar{A}) \cap A$. It is not difficult to show that $\operatorname{Soc}_G(A)$ is an isolated G-invariant subgroup of A.

Let k be a field and let I be an ideal of kA. A subgroup $S_G(I) \leqslant G$ which consists of all elements $g \in G$ such that $I \cap kB = I^g \cap kB$ for some finitely generated dense subgroup $B \leqslant A$ is said to be the standartizer of I in G (see [4]). Since the class $[I]$ consists of all ideals $J \leqslant kA$ such that that $I \cap kB = J \cap kB$ for some finitely generated dense subgroup $B \leqslant A$, we see that $[I]^g = \{J^g \mid J \in [I]\}$ also forms the class $[I^g]$. So, we have an action of G on the set of equivalence classes of ideals of kA. It immediately follows from the definition of $S_G(I)$ that $S_G(I) = S_G([I])$, where $S_G([I]) = \{\gamma \in G \mid [I]^g = [I]\}$ is the stabilizer of $[I]$ in G.

Proposition 3. *Let A be an abelian torsion-free group of finite rank acted by a soluble-by-finite group G, let $C = \operatorname{Soc}_G A$, let k be a field of characteristic zero and let J be an ideal of kA. Let $P_1, P_2, \ldots, P_n$ be ideals of kA such that $\mu_{kB}(J \cap kB) = \{P_i \cap kB \mid 1 \leqslant i \leqslant n\}$ is the set of minimal prime ideals over $J \cap kB$ for some finitely generated dense subgroup B of A. If $|G : S_G([P_i])| < \infty$ for all $1 \leqslant i \leqslant n$ then $J \cap k(C \cap B) \neq 0$.*

Let S be a ring and let M, X and Y be S-modules. The modules X and Y are separated in M if X and Y don't have nonzero isomorphic sections which are isomorphic to a submodule of M. A submodule U of M is said to be essential if $U \cap V \neq 0$ for any nonzero submodule V of M. The module M is said to be uniform if anynonzero submodule of M is essential.

We say that a submodule $X \leqslant M$ is solid in M if X is uniform and M does not have submodules which are isomorphic to a proper section of X.

Let N be a normal subgroup of a group G and let R be a ring. Let M and W be RN-modules. A subgroup $\operatorname{Sep}_{(G,N)}(M,W) \leqslant G$ generated by all elements $g \in G$ such that RN-modules W and Wg are not separated in M is said to be the separator of W in G. Evidently, for any element $h \in G$ such that $h \notin \operatorname{Sep}_{(G,N)}(M,W)$ modules W and $W h$-are separated in M (see [7, 8]).

RN-modules W and V are said to be similar if their injective hulls $[W]$ and $[V]$ are isomorphic. The modules W and V are similar if and only if they have isomorphic essential submodules. By Lemma 3.2 of [9], the stabilizer $\operatorname{Stab}_G[W] = \{g \in G \mid Wg \text{ and } W \text{ are similar}\}$ of W in G is a subgroup of G. It is easy to note that $\operatorname{Stab}_G[W] \leqslant \operatorname{Sep}_{(G,N)}(M,W)$, moreover, if the submodule W is solid in M then $\operatorname{Stab}_G[W] = \operatorname{Sep}_{(G,N)}(M,W)$.

Suppose that the module W and the group G satisfies the conditions of Theorem2 and $\operatorname{Stab}_G[W] = G$. Suppose also that the module W is uniform. Let I be a G-large ideal of kK and A be a torsion-free characteristic central subgroup of finite index in $N/I^\dagger$. By Theorem1, there are a cyclic kN-submodule $0 \neq V \leqslant W$, a G-large ideal I of kK and a central G-invariant subgroup A of finite index in $N/I^\dagger$ such that $M_{kA}(V/VI) = M_{kA}(akN/kNI)$ for any $0 \neq a \in V$ and $M_{kA}(Vg/VgI) = M_{kA}(V/VI)^g$ for any $g \in G$. As the module W is uniform, it is not difficult to note that $\operatorname{Stab}_G[V] = G$ and hence there are nonzero cyclic submodules akN and bkN of V such that $akN \cong (bkN)g$. Then it follows from Theorem 1 that $M_{kA}(V/VI) = M_{kA}(akN/akNI) = M_{kA}((bkN)g/(bkN)gI) = M_{kA}(bkN/bkNI)^g = M_{kA}(V/VI)^g$ and hence $M_{kA}(V/VI) = M_{kA}(V/VI)^g$ for any $g \in G$. So, we have an action of the group G on the finite set $M_{kA}(V/VI)$. By the definition of $M_{kA}(V/VI)$, there are prime ideals $P_1, P_2, \ldots, P_n$ of kA such that $M_{kA}(V/VI) = \{[P_i] \mid 1 \leqslant i \leqslant n\}$. Therefore, as $M_{kA}(V/VI) = M_{kA}(V/VI)^g$ for any $g \in G$, we can conclude that $|G : S_G([P_i])| < \infty$ for all $1 \leqslant i \leqslant n$. Then we can apply Proposition 3 which plays a central role in the proof of the following Theorem.

Theorem 2. *Let N be a nilpotent normal non-abelian minimax torsion-free subgroup of a solvable-by-finite group G of finite torsion-free rank. Let K be a G-invariant subgroup of N such that the quotient group N/K istorsion-free abelian. Let k be a field of characteristic zero. Suppose that there is a uniform kN-torsion kN-module W such that* $\mathrm{Stab}_G[W] = G$ *and for any proper isolated G-invariant subgroup X of N such that* $K \leqslant X$ *the module W is kX-torsion-free. Then* $\mathrm{Soc}_G(N/K) = N/K$.

Let R be a ring and let M be an R-module then $K_R(M)$ denotes the Krull dimension of M. The module M is said to be ρ-critical if $K_R(M) = \rho$ and $K_R(M/U) < K_R(M) = \rho$ for any non-zero submodule $U \leqslant M$.

Let N be a nilpotent group of finite torsion-free rank, let k be a field and let M be an kN-module. Let S be a finitely generated subring of k. Let $0 \neq a \in M$ and let H be a proper isolated normal subgroup of N. We say that (a, H) is an important pair for the SN-module M if there is a finitely generated dense subgroup $A \leqslant N$ such that:

(i) the module aSA is ρ-critical and $K_{SX}(aSA) \leqslant K_{SX}(xSX)$ for any element $0 \neq x \in M$ and any finitely generated dense subgroup $X \leqslant N$;

(ii) $aSA = aSB \otimes_{SB} SA$, where $B = A \cap H$;

(iii) if $0 \neq b \in aSB$ and V is a dense subgroup of B then there is no isolated subgroup $D \leqslant V$ such that $bSV = bSD \otimes_{SD} SV$ and $is_N(D)$ is a normal subgroup of N.

The module M is said to be impervious if it has no important pairs for any finitely generated subring S of k.

Theorem 3. *Let N be a nilpotent non-abelian minimax torsion-free normal subgroup of a soluble-by-finite group G of finite torsion-free rank. Let K be a G-invariant subgroup of N such that the quotient group N/K is torsion-free abelian. Let k be a field of characteristic zero. Let W be a kG-module which is kN-torsion and for any proper isolated G-invariant subgroup X of N such that* $K \leqslant X$ *the module W is kX-torsion-free. Suppose that W is impervious as a kN-module. If* $\mathrm{Sep}_{(G,Y)}(xkG, xkY) = G$ *for any element* $0 \neq x \in W$ *and any G-invariant subgroup Y of N then:*

(i) $\mathrm{Soc}_G(N/K) = N/K$;

(ii) there is a dense G-invariant subgroup $D \leqslant N$ *such that* $K \leqslant D$ *and the quotient group D/K is polycyclic.*

Theorem 4. *Let N be a nilpotent non-abelian minimax torsion-free normal subgroup of a soluble-by-finite group G of finite torsion-free rank. Let k be a field of characteristic zero. Let W be a kG-module which is kN-torsion and for any proper isolated G-invariant subgroup X of N the module W is kX-torsion-free. Suppose that W is impervious as a kN-module and* $\mathrm{Sep}_{(G,Y)}(xkG, xkY) = G$ *for any element* $0 \neq x \in W$ *and any G-invariant subgroup Y of N. Then for any finitely generated subgroup H of G the subgroup N has an H-invariant polycyclic dense subgroup.*

Corollary 1. *Let N be a nilpotent non-abelian minimax torsion-free normal subgroup of a finitely generated linear group G of finite rank. Let k be a field of characteristic zero, let W be a kG-module which is kN-torsion and for any proper isolated G-invariant subgroup X of N the module W is kX-torsion-free. Suppose that W is impervious as a kN-module and* $\mathrm{Sep}_{(G,Y)}(xkG, xkY) = G$ *for any element* $0 \neq x \in W$ *and any G-invariant subgroup Y of N. Then there are a G-invariant polycyclic dense subgroup H of N and an element* $0 \neq a \in W$ *such that:*

(i) $akN = akH \otimes_{kH} kN$ *is a uniform kN-module;*

(ii) the kH-module akH is uniform, impervious and $\mathrm{Stab}_G[akH] = G$.

Impervious modules over group rings of polycyclic groups were considered in [9-12].

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ПРО ДЕЯКІ КОМУТАТИВНІ ІНВАРІАНТИ МОДУЛІВ
НАД МІНІМАКСНИМИ НІЛЬПОТЕНТНИМИ ГРУПАМИ

У статті введено скінченну множину інваріантів для модулів над мінімаксними нільпотентними групами, що складається з класів еквівалентних простих ідеалів групової алгебри абелевої мінімаксної групи. Введена множина інваріантів дає змогу, зокрема, вивчати будову мінімаксної нільпотентної групи N, на якій діє група операторів G, причому N має ін'єктивні модулі, які стабілізуються групою операторів G.

Ключові слова: *нільпотентні групи, мінімаксні групи, групові кільця.*