https://doi.org/10.15407/dopovidi2022.05.010
UDC 517.982.23

K.A. Eftekharinasab, https://orcid.org/0000-0002-4604-3220
Institute of Mathematics of the NAS of Ukraine, Kyiv
E-mail: kaveh@imath.kiev.ua

A multiplicity theorem for Fréchet spaces

Presented by Corresponding Member of the NAS of Ukraine S.I. Maksymenko

This note serves to announce a multiplicity result for Keller C_c^1-functionals on Fréchet spaces which are invariant under the action of a discrete subgroup. For such functionals, we evaluate the minimal number of critical points by applying the Lyusternik—Schnirelmann category.

Keywords: *Fréchet spaces, Lyusternik—Schnirelmann category, Palais—Smale condition, discrete group action.*

We consider a multiplicity problem, namely evaluating the minimal number of the critical orbits of a functional $f : F \to R$ which is invariant under the action of a discrete subgroup G of a Fréchet spaces F. In [1], it was proved that if a functional $f : F \to R$ of the Keller class C_c^1 is bounded from below and satisfies the Palais—Smale condition at the level $c = \inf f$, then c is a critical value for f. Our goal is to significantly improve this result. To this end, we consider functionals which are invariant under a discrete subgroup action. To evaluate the minimal numbers of critical points of such functionals, we employ the Lyusternik–Schnirelmann method.

1. A compactness condition. The initial point of our approach is to introduce a compactness condition of the Palais—Smale type for G-invariant functionals.

We denote by F a Fréchet space whose topology is defined by an increasing sequence of seminorms $(\|\cdot\|_n)$. Moreover, the complete translation-invariant metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

induces the same topology on F. We denote by $B(x, r)$ an open ball with center x and radius $r > 0$ with respect to this metric.

In what follows, we consider only Fréchet spaces over the field R of real numbers. Let E be another Fréchet space, $C(E, F)$ the set of all continuous linear mappings from E to F. A bornology β_E on E is a covering of E satisfying the following:

1. β_E is stable under finite unions;
2. if $A \subseteq \beta_E$ and $B \subseteq A$, then $B \subseteq \beta_E$.

Цитування: Eftekharinasab K.A. A multiplicity theorem for Fréchet spaces. *Допов. Нац. акад. наук Укр.* 2022. № 5. С. 10—15. https://doi.org/10.15407/dopovidi2022.05.010

The compact bornology on E is the family β_{EC} of relatively compact subsets of E having the set of all compact subsets of F as a base, in the sense that every $B \in \beta_{EC}$ is contained in some compact set. We endow the vector space $C(E, F)$ with the β_{EC} -topology which is the topology of uniform convergence on all compact subsets of E. This is a Hausdorff locally convex topology which can be defined by the family of all seminorms obtained as follows:

$$\|L\|_{B,n} = \sup\left\{\|L(e)\|_n : e \in B\right\}$$

where $B \in \beta_{EC}$ and $n \in N$. Let U be an open subset of E and $f : E \to F$ be a mapping. If the directional derivatives

$$f(x)h = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map $df : U \to C(E, F)$ is continuous for all $x \in U$, then we say that f is a Keller C_c^1-mapping (see [2]).

Let $\boldsymbol{L}$ be a topological group with the identity element $\mathbf{e}$. A continuous action of $\boldsymbol{L}$ on F is a mapping $A : \mathbf{L} \times F \to F$, $A(l, m)$ written as $l \cdot m$, such that $\mathbf{e} \cdot m = m$ and $(l_1 * l_2) \cdot m = l_1 \cdot (l_2 \cdot m)$ for all $l_1, l_2 \in \boldsymbol{L}$ and all $m \in F$ (here, * denotes the operation of $\boldsymbol{L}$). A set $A \in F$ is called $\boldsymbol{L}$-invariant, if $l \cdot m \in A$ for all $m \in A$ and all $l \in \boldsymbol{L}$. A functional $f : F \to R$ is called $\boldsymbol{L}$-invariant, if $f(l \cdot m) = f(m)$ for all $l \in \boldsymbol{L}$ and $m \in F$. A mapping $h : F \to F$ is called $\boldsymbol{L}$-equivalent, if $h(l \cdot m) = l \cdot h(m)$ for all $m \in F$ and all $l \in \boldsymbol{L}$. Let G be a discrete subgroup of a Fréchet space F, and let $q : F \to F / G$ be the canonical surjection. A subset $A \subseteq F$ is called q-saturated, if $A = q^{-1} \circ q(A)$. Suppose the space F_1 generated by G has the dimension n. Let F_2 be a topological complement of F_1, such that F is isomorphic to $F_1 \times F_2$. Let $\mathbf{T}^n$ be the n-torus, then $G \simeq Z^n$ and $q(F) \simeq Z^n \times F_2$. Let c be critical point of f. We call the set $q^{-1}(q(c))$ consisting of the critical points of f, a critical orbit of f through c.

Definition 1. Let $f : F \to R$ be a G-invariant functional of the Keller class C_{c}^1. We say that f satisfies the Palais—Smale condition, $\mathrm{PS}_{\mathbf{G}}$-condition for short, if, for every sequence $(x_n) \subset F$ for which $f(x_n)$ is bounded and $f'(x) \to 0$, the sequence $q(x_n)$ contains a convergent subsequence.

2. A multiplicity theorem. To locate critical points, we will apply the strong version of the Ekeland variational principle (see [3]). It states the existence of a certain minimizing sequence on a complete metric space along which we reach the infimum value of the minimization problem.

The Lyusternik—Schnirelmann category $\mathrm{Cat}_T A$ of a subset A of a topological space T is the minimal number of closed sets that cover A and each of which is contractible to a point in T. If $\mathrm{Cat}_T A$ is not finite, we write $\mathrm{Cat}_T A = \infty$. Let $\mathrm{Co}(F)$ be the set of compact subsets of F. Define the sets

$$A_i = \{A \subset F : A \in \mathrm{Co}(T), \mathrm{Cat}_{q(F)} q(A) \geqslant i\}.$$

From [4, Proposition 2.2], it follows that each A_i is a deformation invariant class of subsets of F. The i-th Lyusternik—Schnirelmann minimax value of f is defined by

$$\mu_i = \inf_{A \in A_i} \sup_{x \in A} f(x).$$

The proofs of the following two lemmas are based on the standard arguments, see, for example [4, Lemma 3.2, Lemma 3.3]. Let $CB(F)$ be the family of all nonempty closed and bounded subsets of F. We define the Hausdorff metric on $CB(F)$ by

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

Lemma 1. *The space (A_i, d_H) is a complete metric space.*

Proof. The space $(CB(F), d_H)$ is complete since F is complete (cf. [5]). Thus, we only need to prove that A_i is closed in $CB(F)$. Let $(A_k) \subset A_i$. Suppose $A \in CB(F)$ and $d_H(A_k, A) \to 0$. By [6, Corollary 1.2.13] and [6, Proposition 1.2.14], the space $q(F) \simeq \mathbf{T}^n \times F_2 \simeq (\mathbf{S}^1)^n \times F_2$ is an ANR. So, by [7, Theorem 6.3], there exists a closed neighborhood U of A such that $\mathrm{Cat}_{q(F)}(q(A)) = \mathrm{Cat}_{q(F)} U$. As $q^{-1}(U)$ is a closed neighborhood of the compact set A, there exists k such that $A_k \subset U$. Thereby, $\mathrm{Cat}_{q(F)}(q(A)) = \mathrm{Cat}_{q(F)} U \geqslant \mathrm{Cat}_{q(F)}(q(A_k)) \geqslant i$. Therefore $A \in A_i$.

Lemma 2. *Let $f: F \to R$ be a G-invariant functional of the Keller class C^1_c. Then, the function $\Psi: A_i \to R$ defined by $\Psi(A) = \max_{x\in A} f(x)$ is lower semicontinuous.*

Proof. Let $(B_k) \subset A_i$. Suppose $B \in A_i$ and $d_H(B_k, B) \to 0$. For each $x_0 \in B$, there exists a sequence $(x_j) \subset B_k$ such that $x_j \to x_0$. Thus,

$$f(x_0) = \lim_{j\to\infty} f(x_j) \leqslant \lim_{k\to\infty} \Psi(B_k)$$

and, as $x_0 \in B$ is arbitrary, we have $\psi(B) \leqslant \varliminf_{k\to\infty} \Psi(B_k)$.

Theorem 1. *Let G be a discrete subgroup of a Fréchet spaces F. Assume that the dimension of the space generated by G is a finite number n. Let $f: F \to R$ be a G-invariant functional of the Keller class C^1_c. If f is bounded from below and satisfies the Palais—Smale condition, then f has $n + 1$ critical orbits.*

Proof. Consider the increasing sequence of the Lyusternik—Schnirelmann minimax values μ_i, $1 \leqslant i \leqslant n+1$. Define the sets

$$S_{\mu_i} = \{x \in F : f'(x) = 0, f(x) = \mu_i\}.$$

We claim that if $\mu_i = \mu_k = \mu$ for some $k, i \leqslant k \leqslant n+1$, then S_{μ_i} contains $k - i + 1$ critical orbits. This concludes the proof of the theorem.

We prove the claim by contradiction. Suppose that S_μ contains m distinct critical orbits $q(x_1), \cdots, q(x_m)$ and $m \leqslant k-1$. Pick the positive number r_0 so that, on the balls $B(x_j, 2r_0)$, $1 \leqslant j \leqslant m$ the canonical surjection q is injective. Define the set

$$B_{r_0} = \bigcup_{j=1}^{m} \bigcup_{g\in G} B(x_j + g, r).$$

We show that there exists ε, $0 < \varepsilon^2 < r_0^2$ such that

$$\|f'(x)\|_B > \varepsilon, \quad \forall B \in \beta_{FC} \tag{1}$$

if $x \in f^{-1}([\mu-\varepsilon^2, \mu+\varepsilon^2]) \setminus B_{r_0}$. Because, if (1) is not valid, then there exists a sequence $(x_j) \subset F \setminus B_{r_0}$ such that

$$\left|f(x_j)\right| \leqslant \mu + 1/j \quad \text{and} \quad \|f'(x)\|_{B,n} \leqslant 1/j, \ \forall n \in N, B \in \beta_{FC}.$$

Since f satisfies the $\text{PS}_{\mathbf{G}}$-condition, we may assume that $q(x_j) \to q(\bar{x})$ for some $\bar{x} \in F$. Since f and f' are G-invariant, we may suppose that $x_j \in [0,1]^n \times F_2$.

Whence, $x_j \to \bar{x}$ yields $\bar{x} \in F \setminus B_{r_0}$ and $f(\bar{x}) = \mu$ and $f'(\bar{x}) = 0$ which is impossible, because B_{r_0} is a neighborhood of S_μ. There exists $A \in A_i$ such that

$$\Psi(A) = \max_A f \leqslant \mu + \varepsilon^2.$$

This is achievable by the definition of μ_k. Let $A_0 = A \setminus B_{2r_0}$. By [4, Proposition 2.2], we obtain

$$k \leqslant \text{cat}_{q(F)} q(A) \leqslant \text{cat}_{q(F)}(q(A_0) \cup q(B_{2r_0})) \leqslant \text{cat}_{q(F)} q(A_0) + \text{cat}_{q(F)} q(B_{2r_0}) \leqslant$$

$$\leqslant \text{cat}_{q(F)} q(A_0) + m \leqslant \text{cat}_{q(F)} q(A_0) + k - i.$$

Thus, $A_0 \in A_i$. By Lemma 1, the space (A_i, d_H) is complete. Also, by Lemma 2, the function $\Psi : A_i \to R$ is lower semicontinuous. So, we can employ the Ekeland variational theorem [3, Theorem 4.7]. By the latter theorem, there exists $C \in A_i$ such that

(P1) $\psi(C) \leqslant \psi(A_0) \leqslant \psi(A) \leqslant \mu + \varepsilon^2$,

(P2) $d_H(C, A_0) \leqslant \varepsilon$,

(P3) $\psi(S) > \psi(C) - \varepsilon d_H(C, S), \ \forall S \in A_i, \ S \neq C$.

As $A_0 \cap B_{2r_0} = \varnothing$ and $d_H(C, A_0) \leqslant \varepsilon \leqslant r_0$, then $A \cap B_{2r_0} = \varnothing$. Also, the set $D = \{s \in C : \mu - \varepsilon^2 \leqslant f(s)\}$ is a subset of $f^{-1}([\mu-\varepsilon^2, \mu+\varepsilon^2]) \setminus B_{r_0}$. The set D is closed and, as f is continuous, then it is compact. By (3) for each $y \in D$, there exists $h_{B,y} \in B$ such that

$$\langle f'(y), h_{B,y} \rangle < -\varepsilon. \tag{2}$$

Since f' is continuous, it follows from (2) that there exists $r_y > 0$ such that, for all $g \in G$ and all $h \in F$ with $\|h\|_n < r_y$, we have

$$\langle f'(y+g+h), h_{B,y} \rangle < -\varepsilon.$$

Since D is compact, we can find a subcovering $D_1, \cdots, D_p$ defined by

$$D_i = B(y_i, r_{y_i}).$$

Define the functions $\Phi_i : F \to [0,1]$ by

$$\phi_i(x) = \begin{cases} \dfrac{\sum\limits_{g\in G} d(x+g, \complement D_i)}{\sum\limits_{k=1}^{p}\sum\limits_{g\in G} d(x+g, \complement D_k)}, & x \in \bigcup\limits_{j=1}^{p} D_j, \\ 0, & \textit{otherwise}. \end{cases}$$

Fix a G-invariant continuous function $\Phi : F \to [0,1]$ such that

$$\phi(x) = \begin{cases} 1, & \mu \leqslant f(x), \\ 0, & f(x) \leqslant \mu - \varepsilon^2. \end{cases}$$

Let $r_{\min} = \min\limits_{1\leqslant i\leqslant p} r_{y_i}$. Define the continuous curve $\lambda : [0,1]\times F \to F$

$$\lambda(t,x) = x + tr_{\min}\phi(x)\sum_{i=1}^{p}\psi_i(x)(h_{B,y_i}).$$

For all $x\in F$, all $g \in G$ and all $t\in[0,1]$, we have $\lambda(t, x+g) = \lambda(t,x) + g$.

It follows from [4, Proposition 2.2] that $\operatorname{cat}_{q(F)}(q(\lambda(1,C))) \geqslant \operatorname{cat}_{q(F)}(q(C)) \geqslant i$, whence, as $\lambda(1,C)$ is compact, $\lambda(1,C)\in A_i$. By the mean-value theorem (see [2]) and (P3) for each $y\in D$, there is $T\in(0,1)$ such that

$$f(\lambda(1,C)) - f(x) = \left\langle f'(\lambda(T,C)), r_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)(h_{B,y_i})\right\rangle =$$

$$= r_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)\left\langle f'\left(x + Tr_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)(h_{B,y_i})\right), h_{B,y_i}\right\rangle \leqslant$$

$$\leqslant -\varepsilon r_{\min}\Phi(x).$$

If $x\in D$, then $\Phi(x)=0$ and $f(\lambda(1,x)) = f(x)$.

Let $y_0 \in C$ so that $f(\lambda(1,y_0)) = \psi(D)$. Then, $\mu \leqslant f(\lambda(1,y_0)) - f(y_0) \leqslant -\varepsilon r_{\min}$. So, $y_0\in D$ and $\Phi(y_0)=1$ which imply that $f(\lambda(1,y_0)) - f(y_0) \leqslant -\varepsilon r_{\min}$. Therefore,

$$\psi(S) + \varepsilon d_H(C,S) \leqslant \psi(C).$$

However, $d_H(C,S) \leqslant r_{\min}$ by the definition of S. Hence, $\psi(S) + \varepsilon d_H(C,S) \leqslant \psi(C)$ which contradicts (P3) and concludes the proof.

The author is grateful for the financial support within the program of support for priority research and technical (experimental) developments of the Section of Mathematics of the NAS of Ukraine for 2022-2023; project "Innovative methods in the theory of differential equations, computational mathematics and mathematical modeling", No. 7/1/241 (State Registration No. 0122U000670).

REFERENCES

1. Eftekharinasab, K. (2018). A generalized Palais—Smale condition in the Fréchet space setting. Proc. Int. Geom. Cent., 11, No. 1, pp. 1-11. https://doi.org/10.15673/tmgc.v11i1.915
2. Keller, H. (1974). Differential calculus in locally convex spaces. Berlin: Springer.
3. Hyers, D. H., Isac, G. & Rassias, T. M. (1997). Topics in nonlinear analysis and applications. Singapore: World Scientific.
4. Szulkin, A. (1988). Ljusternik—Schnirelmann theory on C^1-manifolds. Ann. Inst. Henri Poincare (C). Anal. Non Lineaire, 5, Iss. 2, pp. 119-139. https://doi.org/10.1016/S0294-1449(16)30348-1
5. Munkres, J. R. (2000). Topology. 2nd ed. Hoboken: Prentice Hall. Inc.
6. Van Mill, J. (1988). Infinite-dimensional topology. Prerequisites and introduction. Amsterdam: North-Holland.
7. Palais, R. (1966). Lusternik—Schnirelman theory on Banach manifolds. Topology, 5, pp. 115-132. https://doi.org/10.1016/0040-9383(66)90013-9

Received 03.06.2022

К.А. Ефтехарінасаб, https://orcid.org/0000-0002-4604-3220

Інститут математики НАН України, Київ

E-mail: kaveh@imath.kiev.ua

ТЕОРЕМА КРАТНОСТІ ДЛЯ ПРОСТОРІВ ФРЕШЕ

У статті сформульовано теорему кратності для функціоналів з класу Келлера C_c^1 на просторах Фреше. Для таких функціоналів ми даємо мінімальну кількість критичних точок, застосовуючи категорію Люстерника–Шнірельмана.

Ключові слова*: простори Фреше, категорія Люстерника–Шнірельмана, умова Палаіса–Смейла, дія дискретної групи.*