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The Bateman-type variational formalism for an acoustically-driven drop

By employing the Clebsch potentials, the Bateman-type variational formulation for a drop levitating in an acoustic field is proposed when both fluids, liquid drop and external ullage gas, are barotropic, inviscid, compressible and admit rotational flows.

Keywords: *Bateman variational principle, Clebsch potentials, acoustical levitation.*

Idea of the present paper comes from [1, 2] whose objects are two oscillating compressible ideal barotropic fluids when an acoustic vibrator is located in one of them to both generate a high-frequency acoustic field and govern the interface motions. Physically, these two papers deal with an acoustical positioning of a large liquid mass (volume) in microgravity conditions and an acoustically-levitating drop, respectively. *Irrotational* fluid flows are assumed that made it possible to show how to derive the corresponding free-interface boundary value problem based on hydrodynamic variational principles of Hamilton-Ostrogradskii' and Bateman's types. Specifically, the Hamilton-Ostrogradskii principle requires a kinematic constraint but the Bateman's ones derive the complete free-interface boundary value problem. The latter fact makes the Bateman-type principles of especial interest as being important for the multimodal modelling in the liquid sloshing dynamics and, through separation of fast and slow times directly in the action, for deriving a quasi-potential energy functional of the so-called vibro-equilibria, which are time-averaged interfaces between the two fluids that differ in the considered cases from capillary interface shapes governed by gravitation and surface tension.

Assuming *rotational* flows for acoustically-levitating drops can be important due vortices in fluids and/or rotation of the liquid drop itself [3, 4]. This assumption requires a generalization of the Bateman-type variational principles like it has recently been done in [5] for the liquid sloshing problem. Such a generalization is proposed in the present paper.

Throughout the forthcoming text, two compressible barotropic fluids with, possibly, rotational flows, external ullage gas $Q_1(t)$ and liquid drop $Q_2(t)$, are considered in the inertial $Oxyz$ co-

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ordinate frame as illustrated in Fig. 1. Here, $\Sigma(t)$ denotes the unknown *a priori* interface between fluids defined implicitly by the equality $Z(x, y, z, t) = 0$ and the prescribed surface $S(t)$ bounding the ullage gas whose vibrational motions are described by the equality $Y(z, y, z, t) = 0$, where Y is the prescribed function. The outer normal vectors $\boldsymbol{n}$ are determined by $-\nabla Z / |\nabla Z|$ on $\Sigma(t)$ and $-\nabla Y / |\nabla Y|$ on $S(t)$, respectively.

The two fluids are compressible with densities $\rho_1(x, y, z)$ and $\rho_2(x, y, z)$, so that the mass conservation

$$\int\limits_{Q_i(t)} \rho_i dQ = M_i \qquad i = 1, 2 \tag{1}$$

can be treated as geometric constraints; $U(x, y, z) = -\boldsymbol{g} \cdot \boldsymbol{r}$, $\boldsymbol{r} = (x, y, z)$ is responsible for the gravity field.

The velocity fields in $Q_i(t)$ are (non-uniquely [3]) governed by the Clebsch potentials $\varphi_i(x, y, z, t)$, $m_i(x, y, z, t)$, and $\phi_i(x, y, z, t)$, $i = 1, 2$ as follows

$$\boldsymbol{v}_i = \nabla\varphi_i + m_i \nabla\phi_i \,. \tag{2}$$

Based on [6, p. 47], the following Bateman-type Lagrangian is introduced

$$L(\rho_i, \varphi_i, m_i, \phi_i, Z) = -\sum_{i=1}^{2} \int\limits_{Q_i(t)} \rho_i [\partial_t \varphi_i + m_i \partial_t \phi_i + \frac{1}{2} |\boldsymbol{v_i}|^2 + U + E_i(\rho_i)] dQ, \tag{3}$$

where $E_i(\rho_i)$ is the inner energy of the barotropic fluids for which the pressure is postulated by

$$p_i = \rho_i^2 E_i^{'}(\rho_i) \,. \tag{4}$$

The Lagrangian (3) yields the action

$$W(\rho_i, \varphi_i, m_i, \phi_i, Z) = \int\limits_{t_1}^{t_2} \sum_{i=1}^{2} [L(\rho_i, \varphi_i, m_i, \phi_i, Z) - \mu_i M_i] dt \ \text{ for } t_1 < t_2, \tag{5}$$

where $\mu_i = \mu_i(t)$ are the Lagrange multipliers caused by the geometric constraints (1). The action (5) is a function of the fluid densities, the Clebsch potentials and the instant free-interface shape. The zero first variation of the action (5) by $\rho_i, \varphi_i, m_i, \phi_i$, and Z should derive the free-interface boundary value problem, which describes behaviour of the two fluids due to prescribed vibrational motions of $S(t)$.

Remark 1. In contrast to the Bateman-type variational formulation for irrotational flows [1], the acoustical vibrator cannot be determined via the Neuman boundary condition on a fixed gas box surface with anppropriate integral in the Lagrangian (3). One should instead introduce the moving surface $S(t)$.

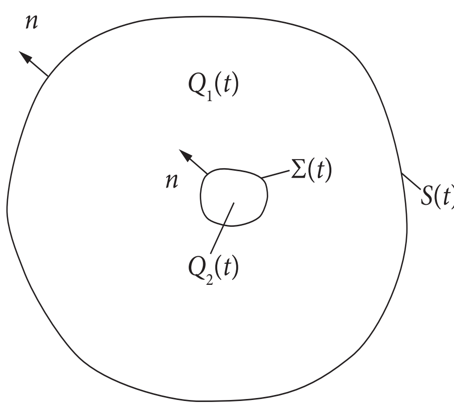

Fig. 1. Schematic sketch of a levitating drop and introduced notations

Henceforth, we *assume* that the Clebsch potentials are smooth functions in $Q_1(t)\bigcup Q_2(t)$. This implies in particular that these functions can be analytically continued through the smooth interface $\Sigma(t)$. Using the calculus of variables, specifically, the Reynolds transport and divergence theorems [7, Appendix A], makes it possible to establish the following propositions.

Lemma 1. *Under the smoothness assumption above, the zero first variation condition*

$$\delta_{\varphi_i} W = 0 \qquad \textit{subject to} \qquad \delta\varphi_i \Big|_{t_1, t_2} = 0 \tag{6}$$

is equivalent to the continuity equation

$$\partial_t \rho_i + \nabla \cdot (\rho_i \boldsymbol{v}_i) = 0 \qquad in \qquad Q_i(t), \tag{7}$$

and the normal-velocity conditions

$$\boldsymbol{v}_i \cdot \boldsymbol{n} = -\frac{\partial_t Z}{|\nabla Z|} \;\; on \;\; \Sigma(t) \qquad (a); \qquad \boldsymbol{v}_1 \cdot \boldsymbol{n} = -\frac{\partial_t Y}{|\nabla Y|} on \quad S(t) \qquad (b). \tag{8}$$

The **proof** is based on the following derivation line with substituting the second condition of (6):

$$-\sum_{i=1}^{2}\int_{t_1}^{t_2}\int_{Q_i(t)} \rho_i [\partial_t \delta\varphi_i + \boldsymbol{v}_i \cdot \nabla(\delta\varphi_i)] dQdt =$$

$$= \int_{t_1}^{t_2}\sum_{i=1}^{2}\left\{ \frac{d}{dt}\int_{Q_i(t)} \rho_i \delta\varphi_i dQ - \int_{Q_i(t)} \delta\varphi_i [\partial_t \rho_i + \nabla\cdot(\rho_i \boldsymbol{v}_i)] dQ + \right.$$

$$\left. +(-1)^i \int_{\sum(t)} \rho_i \left[\frac{\partial_t Z}{|\nabla Z|} + \boldsymbol{v}_i \cdot n \right] \delta\varphi_i dS \right\} + \int_{S(t)} \rho_1 \left[\frac{\partial_t Y}{|\nabla Y|} + \boldsymbol{v}_1 \cdot n \right] \delta\varphi_1 dSdt.$$

Lemma 2. *Under the smoothness assumption above, the zero first variation condition*

$$\delta_{m_i} W = 0 \tag{9}$$

is equivalent to the equations

$$d_t \phi_i = \partial_t \phi_i + \boldsymbol{v}_i \cdot \nabla \phi_i = 0, \tag{10}$$

which implies that the Clebsch potentials ϕ_i remain constant values during motions of liquid particles (the vortex lines move with fluids and always contain the same particles).

The **proof** is based on the expression for this first variation

$$-\sum_{i=1}^{2}\int_{t_1}^{t_2}\int_{Q_i(t)} \rho_i \delta m_i [\partial_t \phi_i + \boldsymbol{v}_i \cdot \nabla\phi_i] dQ\, dt = 0.$$

Lemma 3. *Under the smoothness assumption above and the zero variational condition* (6) *for the action [equivalent to* (7) *and* (8)*], the zero first variation condition*

$$\delta_{\phi_i} W = 0 \qquad \textit{subject to} \qquad \delta\phi_i \big|_{t_1, t_2} = 0 \tag{11}$$

is equivalent to

$$d_t m_i = \partial_t m_i + \boldsymbol{v}_i \cdot \nabla m_i = 0\,, \tag{12}$$

which has the same hydrodynamic meaning like (10) but for the Clebsch potential m_i.

The **proof** uses is the following derivation line

$$-\sum_{i=1}^{2}\int_{t_1}^{t_2}\int_{Q_i(t)} \rho_i [m_i \partial_t \delta\phi_i + \boldsymbol{v}_i \cdot \nabla (m_i \delta\phi_i)] dQdt =$$

$$= -\int_{t_1}^{t_2}\sum_{i=1}^{2}\left\{ \frac{d}{dt}\int_{Q_i(t)} \rho_i m_i \delta\phi_i dQ - \int_{Q_i(t)} \delta\phi_i [\partial_t (m_i\rho_i) + \nabla\cdot(m_i\rho_i\boldsymbol{v}_i)] dQ + (-1)^i \int_{\Sigma(t)} \rho_i m_i \left[\frac{\partial_t Z}{|\nabla Z|} + \boldsymbol{v}_i \cdot n\right]\delta\phi_i dS\right\} +$$

$$+ \int_{S(t)} \rho_1 m_1 \left[\frac{\partial_t Y}{|\nabla Y|} + \boldsymbol{v}_1 \cdot n\right]\delta\phi_1 dSdt$$

together with the second condition of (11) and (8) to show that

$$\partial_t (m_i\rho_i) + \nabla\cdot(m_i\rho_i\boldsymbol{v}_i) = m_i[\partial_t\rho_i + \nabla\cdot(\rho_i\boldsymbol{v}_i)] + \rho_i\ [\partial_t m_i + \boldsymbol{v}_i \cdot \nabla m_i] = 0$$

that, accounting for (7), deduces (12).

Lemma 4. *Under the smoothness assumption above, the zero first variation condition*

$$\delta_{\rho_i} W = 0 \tag{13}$$

is equivalent to the equality

$$\partial_t \varphi_i + m_i \partial_t \phi_i + \frac{1}{2}|\boldsymbol{v}_i|^2 + U + E_i(\rho_i) + \rho_i E_i^{'}(\rho_i) + \mu_i(t) = 0 \quad in \quad Q_i(t)\,, \tag{14}$$

which can be treated as the Bernoulli equation (Lagrange-Cauchy integral) of the Euler equation

$$d_t \boldsymbol{v}_i = \partial_t \boldsymbol{v}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i = -\nabla U - \frac{\nabla p_i}{\rho_i} \quad in \qquad Q_i(t) \tag{15}$$

provided by (9), (11) *and definition* (4).

The **proof** of (14) becomes obvious after taking the variation of (5) by ρ_i. To prove (15), one should apply the gradient operation to equality (14) and definition (4). The second application yields the derivation line

$$\frac{\nabla p_i}{\rho_i} = [2E_i^{'}(\rho_i) + \rho_i E_i^{''}(\rho_i)]\nabla\rho_i = \nabla[E_i(\rho_i) + \rho_i E_i^{'}(\rho_i)].$$

Furthermore, the left-hand side of (15) can be re-written as follows

$$d\,\boldsymbol{v}_i = d\,(\nabla\varphi_i + m_i\nabla\phi_i) = [\nabla\partial_t\varphi_i + m_i\nabla\partial_t\phi_i + \partial_t m_i\nabla\phi_i] + \underbrace{\{\boldsymbol{v}_i\cdot\nabla(\nabla\varphi_i + m_i\nabla\phi_i)\}}_{\boldsymbol{v}_i\nabla\nabla\varphi_i + m_i\boldsymbol{v}_i\cdot\nabla\nabla\phi_i + \nabla\phi_i(\nabla m_i\cdot\boldsymbol{v}_i)} =$$

$$= \nabla\partial_t\varphi_i + m_i\nabla\partial_t\phi_i + \boldsymbol{v}_i\cdot\nabla\nabla\varphi_i + m_i\boldsymbol{v}_i\nabla\nabla\phi_i + \nabla\phi_i[dm_i] \qquad (16)$$

and, applying the gradient operation to the first three quantities in (14) gives

$$\nabla\left(\partial_t\varphi_i + m_i\partial_t\phi_i + \frac{1}{2}|\boldsymbol{v}_i|^2\right) = [\nabla\partial_t\varphi_i + m_i\nabla\partial_t\phi_i + \partial_t\phi_i\nabla m_i] + \boldsymbol{v}_i\nabla\nabla\varphi_i +$$

$$+ m_i\boldsymbol{v}_i\cdot\nabla\nabla\phi_i + \nabla m_i(\nabla\phi_i\cdot\boldsymbol{v}_i) = \nabla\partial_t\varphi_i + m_i\nabla\partial_t\phi_i + \boldsymbol{v}_i\nabla\nabla\varphi_i + m_i\boldsymbol{v}_i\cdot\nabla\nabla\phi_i + \nabla m_i[d\phi_i]. \qquad (17)$$

The right-hand sides in (16) and (17) are identical provided by (10), (12) following from the zero-variation conditions (9) and (11).

Lemma 5. *Under the smoothness assumption above, the zero first variation condition*

$$\delta_Z W = 0 \qquad (18)$$

is equivalent to the interface condition

$$\rho_1[\partial_t\varphi_1 + m_1\partial_t\phi_1 + \frac{1}{2}|\boldsymbol{v}_1|^2 + U + E_1(\rho_1) + \mu_1(t)] =$$

$$= \rho_2[\partial_t\varphi_2 + m_2\partial_t\phi_2 + \frac{1}{2}|\boldsymbol{v}_2|^2 + U + E_2(\rho_2) + \mu_2(t)] \quad on\ \Sigma(t), \qquad (19)$$

which is the same as the traditional dynamic interface condition

$$p_1 = p_2 \ \ on\ \Sigma(t) \qquad (20)$$

provided by (6), (9), (11) *and definition* (5).

Proof. Equality (19) obviously follows from the zero-variation condition (18) by Z:

$$-\int_{t_1}^{t_2}\sum_{i=1}^{2}\int_{Q_i(t)}\rho_i[\partial_t\varphi_i + m_i\partial_t\phi_i + \frac{1}{2}|\boldsymbol{v}_i|^2 + U + E_i(\rho_i) + \mu_i(t)]\frac{(-1)^i\,\delta Z}{|\nabla Z|}dQ = 0.$$

In order to deduce (20), one should note that definition (5) and Bernoulli equation for barotropic compressible fluids (14) derive

$$\rho_i[\partial_t\varphi_i + m_i\partial_t\phi_i + \frac{1}{2}|\boldsymbol{v}_i|^2 + U + E_i(\rho_i) + \mu_i(t)] = -p_i$$

provided, according to conditions of the Lemma 4, by (6), (9), (11).

Summarizing the Lemmas 1-5 shows that the Bateman-type variational formulation derives a free-interface boundary value problem on a drop oscillating in an acoustic field excited by prescribed vibrations of the box surface $S(t)$ as shown in Fig. 1 by consequently applying the necessary condition (6), (9), (11), (13), and (16) to the action (5). The main result can be formulated as the following theorem.

Theorem 6. *Under the smoothness assumption above, the zero first variation of the action* (5),

$$\delta W = \delta_{\varphi_i} W + \delta_{m_i} W + \delta_{\phi_i} W + \delta_{\rho_i} W + \delta_Z W = 0 \ \textit{subject to} \ \delta\varphi_i \big|_{t_1, t_2} = 0 \ \textit{and} \ \delta\phi_i \big|_{t_1, t_2} = 0 \,,$$

is equivalent to the free-interface boundary value problem on acoustically-driven liquid drop $Q_2(t)$ *in ullage gas* $Q_1(t)$ *for a prescribed vibration of the gas box on* $S(t)$. *The differential boundary value problem consists of the continuity equations* (7) *in fluid domains, the kinematic boundary condition* (8a) *on the interface and the `vibrating box surface' condition* (8b), *the Bernoulli equations* (14) *(alternatively, the Euler equations* (15)*) in fluid domains, the dynamic interface condition* (20) *on the interface as well as the vortex line conditions* (10) *and* (12) *provided by the definitions of pressure* (5) *and velocity fields* (2).

Conclusions and discussion. By using the Clebsch potentials, the Bateman-type variational formulation from [1, 2] can be generalised for barotropic fluids to the case of rotational fluid flows. The free-interface boundary value problem derived from the Bateman-type variational formulation not necessary has a unique solution. One should then consider viscous fluid flows.

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ВАРІАЦІЙНИЙ ФОРМАЛІЗМ ТИПУ БЕЙТМЕНА ДЛЯ АКУСТИЧНО КЕРОВАНОЇ КРАПЛІ

Використовуючи потенціали Клебша, пропонується варіаційне формулювання типу Бейтмена для краплі, що левітує в акустичному полі, коли обидві рідини, крапля рідини та зовнішній газ є баротропними, нев’язкими, стисливими та допускають вихорові рухи.

Ключові слова: *віаріаційний принцип Бейтмена, потенціали Клебша, акустична левітація.*